CS191 – Fall 2014
Homework 6 solutions

1. ***Gaussian integral.*** Show that:

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}-i\omega t}d\omega = e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t} \tag{1}$$

We begin by completing the square in the exponent

$$\begin{aligned} & -\frac{(\omega-\omega_0)^2}{2\sigma^2} - i\omega t \\ = & -\frac{1}{2\sigma^2}\left(\omega^2 - 2\omega\omega_0 + i2\omega t\sigma^2 + \omega_0^2\right) \\ = & -\frac{1}{2\sigma^2}\left(\omega^2 - 2\omega(\omega_0 - it\sigma^2) + \omega_0^2 + (-t^2\sigma^4 - i2t\omega_0\sigma^2) - (-t^2\sigma^4 - i2t\omega_0\sigma^2)\right) \\ = & -\frac{(\omega-(\omega_0 - it\sigma^2))^2}{2\sigma^2} - \frac{t^2\sigma^2}{2} - i\omega_0 t \end{aligned}$$

Therefore, the integral becomes:

$$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t}\int_{-\infty}^{\infty} e^{-\frac{(\omega-(\omega_0-it\sigma^2))^2}{2\sigma^2}}d\omega$$

Now, do a change of variables to

$$\Omega = \omega - (\omega_0 - it\sigma^2),$$

under which $d\omega = d\Omega$, but the integration variable has become complex and the limits of integration change to $-\infty + it\sigma^2$ and $\infty + it\sigma^2$. Therefore the integral is:

$$\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t}\int_{-\infty+it\sigma^2}^{\infty+it\sigma^2} e^{-\frac{\Omega^2}{2\sigma^2}}d\Omega = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t}\int_{-\infty}^{\infty} e^{-\frac{\Omega^2}{2\sigma^2}}d\Omega$$

where this equality, where we turned the integral back into one over the real line, is due to integrating over a rectangular contour in the complex plane that goes from $-\infty$ to $+\infty$ to $+\infty + it\sigma$ to $-\infty + it\sigma$, back to $-\infty$. Since the Gaussian function has no singularities, by Cauchy's theorem, the value of this contour integral is zero. In addition the value of the integral along the vertical lines ($+\infty$ to $+\infty + it\sigma$ and $-\infty + it\sigma$ to $-\infty$) is zero, since the Gaussian goes to zero when the real coordinate is $\pm\infty$, this means that

$$\int_{-\infty+it\sigma^2}^{\infty+it\sigma^2} e^{-\frac{\Omega^2}{2\sigma^2}}d\Omega = \int_{\infty}^{-\infty} e^{-\frac{\Omega^2}{2\sigma^2}}d\Omega = \int_{-\infty}^{\infty} e^{-\frac{\Omega^2}{2\sigma^2}}d\Omega \tag{2}$$

This last integral is a standard Gaussian integral that equals $\sqrt{2\pi\sigma^2}$. Therefore, putting this together,

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}-i\omega t}d\omega = e^{-\frac{(t\sigma)^2}{2}-i\omega_0 t} \tag{3}$$

Note: You could also have noted that Eq. (1) defines the Fourier transform of a Gaussian and noted that this results in another Gaussian in the Fourier variable. However, computing this Fourier transform is nontrivial, and is exactly the computation above.

2. ***Generation of the dephasing (or phase-flip) process.*** A general initial state of a qubit has the form

$$\rho_0 = a_0\left|0\right\rangle\left\langle 0\right| + b_0\left|0\right\rangle\left\langle 1\right| + b_0^*\left|1\right\rangle\left\langle 0\right| + c_0\left|1\right\rangle\left\langle 1\right|,$$

with $a_0 + c_0 = 1$. Then evolution for time T by the Hamiltonian

$$H = \frac{\omega}{2}\sigma_z, \tag{4}$$

for a given value of ω yields

$$\begin{aligned}\rho_\omega(T) &= e^{-iHT}\rho_0 e^{iHT} = a_0 e^{-i\frac{\omega}{2}T}\left|0\right\rangle\left\langle 0\right| e^{i\frac{\omega}{2}T} + b_0 e^{-i\frac{\omega}{2}T}\left|0\right\rangle\left\langle 1\right| e^{-i\frac{\omega}{2}T} + b_0^* e^{i\frac{\omega}{2}T}\left|1\right\rangle\left\langle 0\right| e^{i\frac{\omega}{2}T} + c_0 e^{i\frac{\omega}{2}T}\left|1\right\rangle\left\langle 1\right| e^{i\frac{\omega}{2}T}\Big] \\ &= \begin{pmatrix} a_0 & b_0 e^{-i\omega T} \\ b_0^* e^{i\omega T} & c_0 \end{pmatrix}\end{aligned}$$

Now, since ω is a random variable distributed as $\omega \sim \mathcal{N}(0, \sigma^2)$, we must average over its value to get the density matrix under the random Hamiltonian. This averaging is done exactly as it was done in Lecture 14, but now with the mean value $\omega_0 = 0$, to get:

$$\rho(T) = \begin{pmatrix} a_0 & b_0 e^{-\frac{(T\sigma)^2}{2}} \\ b_0^* e^{-\frac{(T\sigma)^2}{2}} & c_0 \end{pmatrix}. \tag{5}$$

This is the density matrix that results from evolution under the uncertain Hamiltonian. We want to compare this to the density matrix that results from the dephasing process:

$$\mathcal{E}(\rho_0) = p\rho_0 + (1-p)\sigma_z\rho_0\sigma_z$$

Computing the output density matrix from this process, for the arbitrary initial state yields:

$$\begin{aligned}\mathcal{E}(\rho_0) &= p\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} + (1-p)\sigma_z\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix}\sigma_z \\ &= p\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} + (1-p)\begin{pmatrix} a_0 & -b_0 \\ -b_0^* & c_0 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & b_0(2p-1) \\ b_0^*(2p-1) & c_0 \end{pmatrix}\end{aligned} \tag{6}$$

We see that the action of the dephasing process is the same as the action of evolution under the uncertain Hamiltonian in Eq. (4). Both lead to a decay of the off-diagonal elements. In particular, equating Eq. (5) and Eq. (6) gives us:

$$2p - 1 = e^{-\frac{(T\sigma)^2}{2}} \quad \Rightarrow \quad p = \frac{1 + e^{-\frac{(T\sigma)^2}{2}}}{2}$$

Note that this $\frac{1}{2} \leq p \leq 1$.

3. ***Properties of the Lindblad master equation.*** Prove that the Lindblad master equation,

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[h_0 + h_{LS}, \rho(t)] + \sum_{k=1}^{K}\gamma_k\left(L_k\rho(t)L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho(t) - \frac{1}{2}\rho(t)L_k^\dagger L_k\right),$$

preserves the trace and Hermiticity of $\rho(t)$.

(a) To prove that this evolution preserves Hermiticity, we need to show that $\rho(t+dt) = \rho(t) + dt\left(\frac{d}{dt}\rho(t)\right)$ is Hermitian if $\rho(t)$ is. To do this, take the Hermitian conjugate of $\rho(t+dt)$:

$$\begin{aligned}\rho(t+dt)^\dagger &= \rho(t)^\dagger + dt\left(-\frac{i}{\hbar}[h_0 + h_{LS}, \rho(t)] + \sum_{k=1}^{K}\gamma_k\left(L_k\rho(t)L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho(t) - \frac{1}{2}\rho(t)L_k^\dagger L_k\right)\right)^\dagger \\ &= \rho(t) + dt\left(+\frac{i}{\hbar}[\rho(t), h_0 + h_{LS}] + \sum_{k=1}^{K}\gamma_k\left(\left[L_k\rho(t)L_k^\dagger\right]^\dagger - \frac{1}{2}\left[L_k^\dagger L_k\rho(t)\right]^\dagger - \frac{1}{2}\left[\rho(t)L_k^\dagger L_k\right]^\dagger\right)\right) \\ &= \rho(t) + dt\left(-\frac{i}{\hbar}[h_0 + h_{LS}, \rho(t)] + \sum_{k=1}^{K}\gamma_k\left(L_k\rho(t)L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho(t) - \frac{1}{2}\rho(t)L_k^\dagger L_k\right)\right) \\ &= \rho(t+dt).\end{aligned}$$

Here we have used the following facts about the Lindblad master equation: $\rho(t), h_0$ and h_{LS} are Hermitian, and γ_k are real.

(b) To prove that the Lindblad master equation preserves trace we need to show that $\mathrm{tr}\,(\rho(t+dt) = \mathrm{tr}\,(\rho(t))$. To do this, evaluate the trace of $\rho(t+dt)$:

$$\begin{aligned}\mathrm{tr}\,(\rho(t+dt)) &= \mathrm{tr}\,(\rho(t)) + dt\left\{-\frac{i}{\hbar}\mathrm{tr}\,([h_0 + h_{LS}, \rho(t)])\right.\\ &\quad\left.+\sum_{k=1}^{K}\gamma_k\left(\mathrm{tr}\,(L_k\rho(t)L_k^\dagger) - \frac{1}{2}\mathrm{tr}\,(L_k^\dagger L_k\rho(t)) - \frac{1}{2}\mathrm{tr}\,(\rho(t)L_k^\dagger L_k)\right)\right\}.\end{aligned}$$

Now, using the cyclic property of trace (*i.e.*, $\mathrm{tr}\,(ABC) = \mathrm{tr}\,(CAB) = \mathrm{tr}\,(BCA)$), we see that the quantity multiplying dt, in the curly braces, is zero. Therefore,

$$\mathrm{tr}\,(\rho(t+dt)) = \mathrm{tr}\,(\rho(t))$$

as required.

4. ***Action of channels on the Bloch vector.***

(a) Phase-flip channel. We computed the output of the phase flip channel for a general state in question 2 above. It is:

$$\begin{pmatrix} a_0 & b_0(2p-1) \\ b_0^*(2p-1) & c_0 \end{pmatrix}$$

The Bloch vector elements for this resulting density matrix are:

$$\begin{aligned} x &= 2\mathrm{Re}\{b\} = 2\mathrm{Re}\{b_0(2p-1)\} = (2p-1)2\mathrm{Re}\{b_0\} \\ y &= -2\mathrm{Im}\{b\} = -2\mathrm{Im}\{b_0(2p-1)\} = (2p-1)(-2\mathrm{Im}\{b_0\}) \\ z &= a - c = a_0 - c_0 \end{aligned}$$

Therefore this process transforms the Bloch vector as:

$$(x_0, y_0, z_0) \rightarrow (x_0(2p-1), y_0(2p-1), z_0).$$

Remember that $|2p-1| \leq 1$. Therefore this process shrinks the x and y components of the Bloch vector, while leaving the z component unchanged.

(b) Bit-flip channel. The output of the bit-flip channel for an arbitrary input state is:

$$\begin{aligned}\mathcal{E}(\rho_0) &= p\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} + (1-p)\sigma_x\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix}\sigma_x \\ &= p\begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} + (1-p)\begin{pmatrix} c_0 & b_0^* \\ b_0 & a_0 \end{pmatrix} \\ &= \begin{pmatrix} c_0 + p(a_0 - c_0) & b_0^* - p(b_0^* - b_0) \\ b_0 + p(b_0^* - b_0) & a_0 - p(a_0 - c_0) \end{pmatrix}\end{aligned}$$

Now, let us look at the Bloch vector components of this resulting density matrix:

$$\begin{aligned} x &= 2\mathrm{Re}\{b\} = 2\mathrm{Re}\{b_0^* - p(b_0^* - b_0)\} = 2\mathrm{Re}\{b_0^* - i2p\mathrm{Im}\{b_0^*\}\} = 2\mathrm{Re}\{b_0\} \\ y &= -2\mathrm{Im}\{b\} = -2\mathrm{Im}\{b_0^* - p(b_0^* - b_0)\} = -2\mathrm{Im}\{\mathrm{Re}\{b_0\} - i\mathrm{Im}\{b_0\} - i2p\mathrm{Im}\{b_0^*\}\} = (2p-1)(-2\mathrm{Im}\{b_0\}) \\ z &= a - c = c_0 + p(a_0 - c_0) - a_0 + p(a_0 - c_0) = (2p-1)(a_0 - c_0) \end{aligned}$$

Therefore, the Bloch vector transforms as:

$$(x_0, y_0, z_0) \rightarrow (x_0, y_0(2p-1), z_0(2p-1)).$$

This process shrinks the y and z components of the Bloch vector while keeping the x component unchanged.

(c) Depolarizing channel. The output of the depolarizing channel for an arbitrary input state is:

$$
\begin{aligned}
\mathcal{E}(\rho_0) &= p \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + (1-p) \begin{pmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{p}{2} + (1-p)a_0 & (1-p)b_0 \\ (1-p)b_0^* & \frac{p}{2} + (1-p)c_0 \end{pmatrix}
\end{aligned}
$$

The Bloch vector elements for this output density matrix are

$$
\begin{aligned}
x &= 2\text{Re}\{b\} = 2\text{Re}\{(1-p)b_0\} = (1-p)2\text{Re}\{b_0\} \\
y &= -2\text{Im}\{b\} = -2\text{Im}\{(1-p)b_0\} = (1-p)(-2\text{Im}\{b_0\}) \\
z &= a - c = \frac{p}{2} + (1-p)a_0 - \frac{p}{2} - (1-p)c_0 = (1-p)(a_0 - c_0)
\end{aligned}
$$

Therefore the Bloch vector transforms as:

$$
(x_0, y_0, z_0) = (x_0(1-p), y_0(1-p), z_0(1-p))
$$

Thus the depolarizing process uniformly shirks all the elements of the Bloch vector by the factor $(1-p)$.