CS191 – Fall 2014
Homework 7 solutions

1. ***Error correction for a mixture of errors***. Suppose $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$ is a general single qubit state encoded in the bit flip code. Then, due to errors it is mapped to the following mixed state:

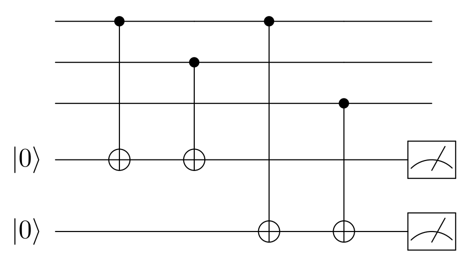
$$\rho = (1-p)\rho_0 + \frac{p}{3}(X_1\rho_0X_1 + X_2\rho_0X_2 + X_3\rho_0X_3),$$

where $\rho_0 = |\psi\rangle\langle\psi|$, and $X_1 = \sigma_x \otimes I \otimes I$ and so on.

(a) We write out the erroneous state in braket form as:

$$\begin{aligned}\rho &= (1-p)\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big)\\ &+ \frac{p}{3}\Big(|\alpha|^2|100\rangle\langle 100| + \alpha\beta^*|100\rangle\langle 011| + \alpha^*\beta|011\rangle\langle 100| + |\beta|^2|011\rangle\langle 011|\\ &\quad + |\alpha|^2|010\rangle\langle 010| + \alpha\beta^*|010\rangle\langle 101| + \alpha^*\beta|101\rangle\langle 010| + |\beta|^2|101\rangle\langle 101|\\ &\quad + |\alpha|^2|001\rangle\langle 001| + \alpha\beta^*|001\rangle\langle 110| + \alpha^*\beta|110\rangle\langle 001| + |\beta|^2|110\rangle\langle 110|\Big)\end{aligned}$$

(b) Compute the state that is produced when the bit-flip code error detection circuit



is executed with the state of the first three qubits being ρ. What are the probabilities of getting the four possible measurement results (00, 01, 10, and 11) when the ancilla are measured?
The circuit specifies a unitary to be applied to the input state $\rho^0_{SA} = \rho \otimes |00\rangle\langle 00|$, where the tensor product separates the input state ρ and the ancilla qubits. The state after the CNOTs is:

$$\begin{aligned}&(1-p)\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big) \otimes |00\rangle\langle 00|\\ +& \frac{p}{3}\Big(|\alpha|^2|100\rangle\langle 100| + \alpha\beta^*|100\rangle\langle 011| + \alpha^*\beta|011\rangle\langle 100| + |\beta|^2|011\rangle\langle 011|\Big) \otimes |11\rangle\langle 11|\\ +& \frac{p}{3}\Big(|\alpha|^2|010\rangle\langle 010| + \alpha\beta^*|010\rangle\langle 101| + \alpha^*\beta|101\rangle\langle 010| + |\beta|^2|101\rangle\langle 101|\Big) \otimes |10\rangle\langle 10|\\ +& \frac{p}{3}\Big(|\alpha|^2|001\rangle\langle 001| + \alpha\beta^*|001\rangle\langle 110| + \alpha^*\beta|110\rangle\langle 001| + |\beta|^2|110\rangle\langle 110|\Big) \otimes |01\rangle\langle 01|\end{aligned}$$

When we measure the ancilla qubits, there are four possible results, corresponding to the post-measurement states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. The probabilities for each outcome are:

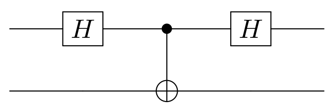
$$\begin{aligned}\Pr\{00\} &= 1-p\\ \Pr\{11\} &= \frac{p}{3}\\ \Pr\{10\} &= \frac{p}{3}\\ \Pr\{01\} &= \frac{p}{3}\end{aligned}$$

(c) After application of the corresponding correction gates, we get the states:

$$\begin{aligned}
&\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big) \otimes |00\rangle\langle 00| \quad \text{with probability} \quad (1-p)\\
&\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big) \otimes |11\rangle\langle 11| \quad \text{with probability} \quad \frac{p}{3}\\
&\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big) \otimes |10\rangle\langle 10| \quad \text{with probability} \quad \frac{p}{3}\\
&\Big(|\alpha|^2|000\rangle\langle 000| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000| + |\beta|^2|111\rangle\langle 111|\Big) \otimes |01\rangle\langle 01| \quad \text{with probability} \quad \frac{p}{3}
\end{aligned}$$

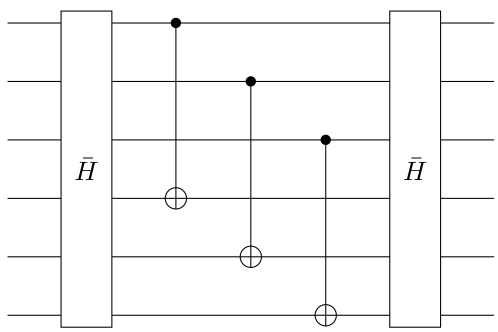
So we see that the error-free state of the encoded qubit is recovered regardless of which ancilla state is measured. Even though the error process resulted in a probabilistic mixture of errors, the measurement of the error syndrome (the ancilla values after the error detection circuit) results in only one of the possible errors being realized, and this error can then be corrected (as long as each error in the the original mixture of errors is correctable by the code being used).

2. ***Bit flip code encoding and logical evolution***. Suppose we want to execute the following circuit



where the two input states are arbitrary single qubit states. But we are worried about bit flip errors occurring during the circuit and so we encode each qubit in the bit flip code.

(a) We did not cover the form of the transversal Hadamard in class, so let us just denote it with a block. Hence this logical circuit once the qubits are encoded in the bit flip code is:



(b) Let us first compute what the result of the circuit should be if there are no errors. The error free initial state is $|000\rangle \otimes |000\rangle$. The Hadamard gate is $\frac{1}{\sqrt{2}}(X+Z)$, where X and Z are the Pauli operators. So using the logical operators for the bit flip code (which we went over in class, the logical Hadamard operator for the bit flip code becomes

$$\bar{H} = \frac{1}{\sqrt{2}}(XXX + ZZZ)$$

So the (unnormalized) state after the logical Hadamard is

$$((XXX + ZZZ)|000\rangle) \otimes |000\rangle = (|000\rangle + |111\rangle) \otimes |000\rangle$$

Then doing the CNOTs, we get

$$|000\rangle \otimes |000\rangle + |111\rangle \otimes |111\rangle$$

Then doing the logical Hadamard again results in (the unnormalized state)

$$\begin{aligned}
&(|000\rangle + |111\rangle) \otimes |000\rangle + (|000\rangle - |111\rangle) \otimes |111\rangle\\
&= (|\bar{0}\rangle + |\bar{1}\rangle) \otimes |\bar{0}\rangle + (|\bar{0}\rangle - |\bar{1}\rangle) \otimes |\bar{1}\rangle
\end{aligned} \tag{1}$$

where the overhead bar indicates an encoded logical state. So Eq. (1) is the correct output of the encoded circuit.

Now let us see what happens to the erroneous input state $|100\rangle \otimes |000\rangle$. First, applying the logical Hadamard results in the unnormalized state

$$((XXX + ZZZ)|100\rangle) \otimes |000\rangle = (|011\rangle - |100\rangle) \otimes |000\rangle$$

Then applying the CNOTs we get,

$$|011\rangle \otimes |011\rangle - |100\rangle \otimes |100\rangle$$

Finally, applying the logical Hadamard again results in

$$\begin{aligned} &(|100\rangle + |011\rangle) \otimes |011\rangle - (|011\rangle - |100\rangle) \otimes |100\rangle \\ &= |100\rangle \otimes |011\rangle + |011\rangle \otimes |011\rangle - |011\rangle \otimes |100\rangle + |100\rangle \otimes |100\rangle \end{aligned}$$

Now the error detection circuit checks the parity of the first block and the parity of the second block. Note that in both cases the parity indicates that the first qubit is in error (verify this by application of the error detection circuit if you need to). Therefore the correction step will flip the first qubit in block 1 and the first qubit in block 2 to obtain:

$$\begin{aligned} &|000\rangle \otimes |111\rangle + |111\rangle \otimes |111\rangle - |111\rangle \otimes |000\rangle + |000\rangle \otimes |000\rangle \\ &(|\bar{0}\rangle + |\bar{1}\rangle) \otimes |\bar{1}\rangle + (|\bar{0}\rangle - |\bar{1}\rangle) \otimes |\bar{0}\rangle \end{aligned} \tag{2}$$

Now compare the result of the error free circuit evaluation, Eq. (1), and the circuit evaluation with the erroneous input state, Eq. (2). The latter can be seen as either, (i) a logical phase flip error on the first block, or (ii) a logical bit flip error on the second block. Either way this *logical* error slipped by the error detection and correction steps we applied, and was thus uncorrectable using the code we used (the bit flip code).

Notice what happened here. We used a code that can correct at most one bit flip error, and only one bit flip error occurred in the whole process. However, even so the output state is incorrect. This shows that one must use a full quantum code (capable of correcting bit flip and phase flip errors) even when a limited set of errors can happen. The logical gates one wants to perform can turn one type of error (e.g. a bit flip error) into other types (e.g. a phase flip error). Therefore, one requires a full code (e.g. the Shor code) for general computations.