C 191 - FALL 2014
Homework 8: due in lecture Nov. 19

1. *Unitarity of Fourier Transform*

   The (classical) Fourier transform mod N is the $N \times N$ matrix given by

$$FT_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix} ,$$

   where $\omega = e^{2\pi i/N}$ is a primitive Nth root of unity. So the i,j'th element of FT_N is $\frac{1}{\sqrt{N}}\omega^{ij}$, for $i,j = 0,\ldots,N-1$.

   Show that FT_N is unitary by evaluating the inner product between the ith and jth columns of FT_N, i.e., show that $\langle i | FT_N^\dagger FT_N | j \rangle = \delta_{ij} \equiv \left\{ \begin{smallmatrix} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{smallmatrix} \right.$.

   Your calculation will demonstrate a very important and useful result, namely that for ω a primitive Nth root of unity (i.e., $\omega^N = 1$ but $\omega^m \neq 1$ for $0 < m < N$),

$$\sum_{k=0}^{N-1} \omega^{kj} = \begin{cases} N & \text{if } j = 0 \mod N \\ 0 & \text{otherwise} \end{cases} .$$

2. *Fourier Transforms and the uncertainty principle*

   The uncertainty principle bounds how well a quantum state can be localized simultaneously in the standard basis and the Fourier basis. In this question, we will derive an uncertainty principle for a discrete system of n-qubit.

   Let $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$ be the state of an n-qubit system. A measure of the spread of $|\psi\rangle$ is $S(|\psi\rangle) \equiv \sum_x |\alpha_x|$. For example, for a completely localized state $|\psi\rangle = |y\rangle$ ($y \in \{0,1\}^n$), the spread is $S(|\psi\rangle) = 1$. For a maximally spread state $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_x |x\rangle$, $S(|\psi\rangle) = 2^n \cdot \frac{1}{\sqrt{2^n}} = \sqrt{2^n}$.

   a) Prove that for any quantum state $|\psi\rangle$ on n qubits, $S(|\psi\rangle) \leq 2^{n/2}$. (Hint: use the Cauchy-Schwarz inequality,$\langle v | w \rangle \leq \|v\| \cdot \|w\|$.)

   b) Suppose that $|\alpha_x| \leq a$ for all x. Prove that $S(|\psi\rangle) \geq \frac{1}{a}$. (Hint: think about normalization....)

   c) Show that $H^{\otimes n}|x\rangle = \sum_y (-1)^{x\cdot y}|y\rangle$ ($x \cdot y \equiv \sum_{i=1}^n x_i y_i$).

   d) Using c), the action of $H^{\otimes n}$ on $|\psi\rangle$ can be written as $H^{\otimes n}|\psi\rangle = \sum_x \beta_x |x\rangle$, where $\beta_x = \frac{1}{2^{n/2}} \sum_y (-1)^{x\cdot y} \alpha_y$.

   Use this to prove that for all y, $|\beta_y| \leq \frac{1}{2^{n/2}} S(|\psi\rangle)$. (Hint: use the triangle inequality.)

   d) Prove the uncertainty relation $S(|\psi\rangle) S(H^{\otimes n}|\psi\rangle) \geq 2^{n/2}$. Justify why it makes sense to call this an uncertainty relation.