C191 - Lecture 1

I. AN INCOMPLETE LIST OF THE AXIOMS OF QUANTUM MECHANICS

1. Every physical system is associated with a Hilbert space. A Hilbert space is a vector space (collection of objects that can be added together and multiplied by scalars) together with an inner product, $\langle a|b\rangle$ such that:
   (a) Complex conjugate relation: $\langle a|b\rangle = \langle b|a\rangle^*$
   (b) Linear: $\langle x_1 a_1 + x_2 a_2|b\rangle = x_1 \langle a_1|b\rangle + x_2 \langle a_2|b\rangle$
   (c) Positive definite: $\langle a|a\rangle \geq 0$
2. Every ray in the Hilbert space is associated with a state of the system. We sometimes call these rays *kets* - and give them labels such as, $|\psi\rangle$. Dual vectors are represented by *bras* and are labeled $\langle\psi|$. In the matrix representations, kets are column vectors and the bras are the conjugate transpose of the kets, making them row vectors.
3. Each measurement corresponds to Hermitian operator, $A = A^\dagger \equiv A^{\mathsf{T}*}$.
4. The possible results of the measurements are the eigenvalues of A.
5. The probability of observing a particular value is given by $\text{Prob}(a = a_n) = \frac{\langle\psi|P_{a_n}|\psi\rangle}{\langle\psi|\psi\rangle}$, where P_{a_n} is the projector onto the set of states which correspond to given measurement value (this is also known as the a_n eigenspace of A). The denominator in this expression is to handle the case where the state vector has not been properly normalized.

II. STATES

Let's say we have a three-dimensional Hilbert space spanned by the orthogonal basis vectors, $|0\rangle, |1\rangle, |2\rangle$. Because they are orthogonal, their inner products are: $\langle 0|1\rangle = \langle 0|2\rangle = \langle 1|2\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = \langle 2|2\rangle = 1$. One particular state in this Hilbert space is

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ i/\sqrt{3} \end{pmatrix}$$

This state is an example of *superposition*, where the state of the system is a sum of basis states. We have written the state using the bra-ket notation as well as using a column-vector representation of the state where the entries in the column vector correspond to the coefficients in front of the basis vectors in the bra-ket notation. The coefficients in front of the basis vectors are called *amplitudes*, and the probability of finding the system in any given basis state is given by the modulus squared of the amplitude of this basis vector. In this case the probabilities are all 1/3 - you should check this! That quantum mechanics predicts probabilities is the first major differences between it and classical mechanics. In classical mechanics, complete knowledge of the state of the system could tell you exactly what the result of any measurement would be. But quantum mechanics is fundamentally probabilistic and there does not appear to be any underlying deterministic theory.

Note that could also determine these probability using the axioms above. For instance - what is the probability of finding the system in the $|0\rangle$ state? The fifth axiom says that the probability is given in terms of a projector (we'll ignore the denominator since the state is normalized: $\langle\psi|\psi\rangle = 1$), and the the projector onto the $|0\rangle$ state is the operator:

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while the *bra* vector for the state is

$$\langle\psi| = |\psi\rangle^\dagger = \frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2| = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3} \end{pmatrix}$$

We can compute the inner product $\mathrm{Prob}(0) = \langle\psi|P_0|\psi\rangle = \langle\psi\,|0\rangle\langle 0|\,\psi\rangle = |\langle 0|\psi\rangle|^2$ using either matrix notation:

$$\begin{aligned}\mathrm{Prob}(0) &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\i/\sqrt{3}\end{pmatrix}\\ &= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3}\end{pmatrix}\begin{pmatrix}1/\sqrt{3}\\0\\0\end{pmatrix}\\ &= 1/3\end{aligned}$$

or with braket notation:

$$\begin{aligned}\mathrm{Prob}(0) &= \left(\frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2|\right)(|0\rangle\langle 0|)\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle\right)\\ &= \left(\frac{1}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1| + \frac{-i}{\sqrt{3}}\langle 2|\right)\left(\frac{1}{\sqrt{3}}|0\rangle\right)\\ &= 1/3\end{aligned}$$

We'll be very concerned with collections of two-level systems (qubits) from now on. With a single qubit, we usually label the states as $|0\rangle$ and $|1\rangle$. These states comprise the *standard* or *computational* basis. Another important basis is the *plus-minus* basis defined as $|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$.

A. Combining quantum systems

One feature of quantum systems that makes them (i) very difficult to simulate and (ii) powerful computationally is what happens when many small systems are combined. Consider a collection of three one-qubit Hilbert spaces, $\mathcal{H}_1$, each spanned by the vectors $\{|0\rangle, |1\rangle\}$. The Hilbert space of the combined system, denoted $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_1$ or $\mathcal{H}_1^{\otimes 3}$, is spanned by the vectors: $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$. This space is therefore eight-dimensional. For each qubit that is added to the system, the dimension doubles. Simulating the exact dynamics of a collection of 30 qubits requires our keeping track of $2^{30} \simeq 10^9$ complex numbers (one amplitude for each basis vector). This combination is called a *tensor product*. The tensor product is linear, so

$$\begin{aligned}(a|0\rangle + b|1\rangle)\otimes(c|0\rangle + d|1\rangle) &= ac|0\rangle\otimes|0\rangle + ad|0\rangle\otimes|1\rangle + bc|1\rangle\otimes|0\rangle + bd|1\rangle\otimes|1\rangle\\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle\end{aligned}$$

Here for convenience we've written, for example, $|0\rangle\otimes|0\rangle = |00\rangle$, the tensor product notation as being implied.

Note that this is different from classical systems, which combine by *direct sum*, where the dimension of the resulting space is the *sum* of the constituent spaces. This can be understood by considering a collection of gas molecules. For each particle, we need only keep track its position (3 real numbers) and its velocity (3 real numbers). Adding a particle to our collection then requires us to keep track of an additional 6 terms. Simulating classical mechanics is a problem whose difficulty increases only like a polynomial in the number of particles, whereas simulating quantum mechanics is a problem whose difficulty increases exponentially in the number of particles. We'll see this separation of polynomial vs exponential difficulty later in the course.

B. Measuring qubits in different bases

Suppose we have a state specified in the standard basis

$$|\phi\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

the probability of measuring 0 or 1 can be read off directly as 2/3 and 1/3, respectively. But suppose we had a measurement device which could only distinguish between states in the plus-minus basis? What then is the probability of each? There are a few ways to proceed. Firstly, we could rewrite the state in the plus minus basis and read off the coefficients. From the definition of the plus-minus basis above, we see that $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}-$ and $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}-$. Therefore,

$$
\begin{aligned}
|\phi\rangle &= \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \\
&= \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) \\
&= \left(\sqrt{\frac{2}{6}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\sqrt{\frac{2}{6}} - \frac{1}{\sqrt{6}}\right)|-\rangle
\end{aligned}
$$

So the probabilities of measuring + and - are $\left(\sqrt{\frac{2}{6}} + \frac{1}{\sqrt{6}}\right)^2 \simeq 0.97$ and $\left(\sqrt{\frac{2}{6}} - \frac{1}{\sqrt{6}}\right)^2 \simeq 0.03$ respectively. Alternatively, we can use the axioms. For projection onto a single state, we saw that we simply need to take the modulus squared of the inner product of the given state with the target state. So:

$$
\begin{aligned}
\text{Prob}_+ &= |\langle +|\phi\rangle|^2 \\
&= \left|\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\langle 0|1\rangle + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\langle 1|0\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\langle 1|1\rangle\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} \times 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \times 0 + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} \times 0 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \times 1\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\right|^2 \\
&\simeq 0.97
\end{aligned}
$$