Introduction to Quantum Error Correction

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Quantum Information and Quantum Computation, CUP 2000, Ch. 10

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Errors in QIP

- unitary $\alpha|0\rangle+\beta|1\rangle \xrightarrow{U} \alpha|0\rangle+\beta e^{i(\phi+\delta)}|1\rangle$
- non-unitary $\alpha|0\rangle+\beta|1\rangle \xrightarrow{M} p_\alpha|0\rangle$
- general: pure $\rightarrow$ mixed states

$$|\psi\rangle \rightarrow \rho_f \qquad tr\rho_f^{\,2} < 1$$

$$\rho = |\psi\rangle\langle\psi| \rightarrow \rho_f = \sum_k E_k \rho E_k^\dagger = \sum_k E_k |\psi\rangle\underbrace{\langle\psi|E_k^\dagger}_{|\psi_k\rangle}$$

from $\rho_f = tr_{env}\left[U\rho\otimes\rho_{env}U^\dagger\right]$

$$= \sum_k \langle e_k|U\rho\otimes|e_0\rangle\langle e_0|U^\dagger|e_k\rangle$$

$$= \sum_k E_k\rho E_k^\dagger \quad E_k = \langle e_k|U|e_0\rangle,\ U(\text{sys+env})$$

trace preserving: $\sum_{\mathrm{k}} E_k E_k^\dagger = 1$

$\equiv$ take ρ and randomly replace by $E_k\rho E_k^\dagger = |\psi_k\rangle\langle\psi_k|$

with probability $p_k = tr\left(E_k\rho E_k^\dagger\right)$

Quantum noise:

channel representation

$$\rho \to \sum_k E_k \rho E_k^\dagger$$

Bit flip channel

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_1 = \sqrt{1-p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Phase flip channel

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_1 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Bit-phase flip channel

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_1 = \sqrt{1-p}Y = \sqrt{1-p}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Amplitude damping channel

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma^2} \end{pmatrix}, E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Depolarizing channel

$$\varepsilon(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

Geometrical interpretation: Bloch sphere in r-space (NC p. 376)

$$\rho = \frac{I + \overline{r}\cdot\overline{\sigma}}{2}, \overline{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}, \overline{r} = \{r_x, r_y, r_z\}$$

Repetition codes

classical $\quad 0 \rightarrow 000$
$1 \rightarrow 111$

error, e.g. 010, corrected to majority value $\rightarrow$ 000
note: learned value of bits in doing so

prob. for bit error $p < 1$:

multi-bit error prob. $= 3p^2(1-p)+p^3=3p^2-2p^3$
$< p$ when $p < 0.5$

$0 \rightarrow 00000\ldots$ n bits, majority $n/2+1$
$1 \rightarrow 11111\ldots$ $\Rightarrow$ error prob. $\cong p^{n/2+1} + \ldots$
$\Rightarrow$ error prob. $\downarrow$ as n $\uparrow$ ($p < 0.5$)

quantum? $|\psi\rangle \xrightarrow{?} |\psi\rangle|\psi\rangle|\psi\rangle$

No cloning theorem!

suppose $|\psi\rangle \rightarrow |\psi\rangle|\psi\rangle$ and $|\phi\rangle \rightarrow |\phi\rangle|\phi\rangle$

then $(|\psi\rangle+|\phi\rangle) \rightarrow (|\psi\rangle+|\phi\rangle)(|\psi\rangle+|\phi\rangle)$

$$= |\psi\psi\rangle + |\phi\phi\rangle + |\phi\psi\rangle + |\psi\phi\rangle$$

but $|\psi\rangle+|\phi\rangle \rightarrow |\psi\psi\rangle+|\phi\phi\rangle$ by linearity

cannot copy <u>unknown</u> quantum states

Encode/Error/Recovery

- quantum information is encoded into ρ_C
- an error occurs

$$\varepsilon(\rho_C) = \sum_k E_k \rho_C E_k^\dagger$$

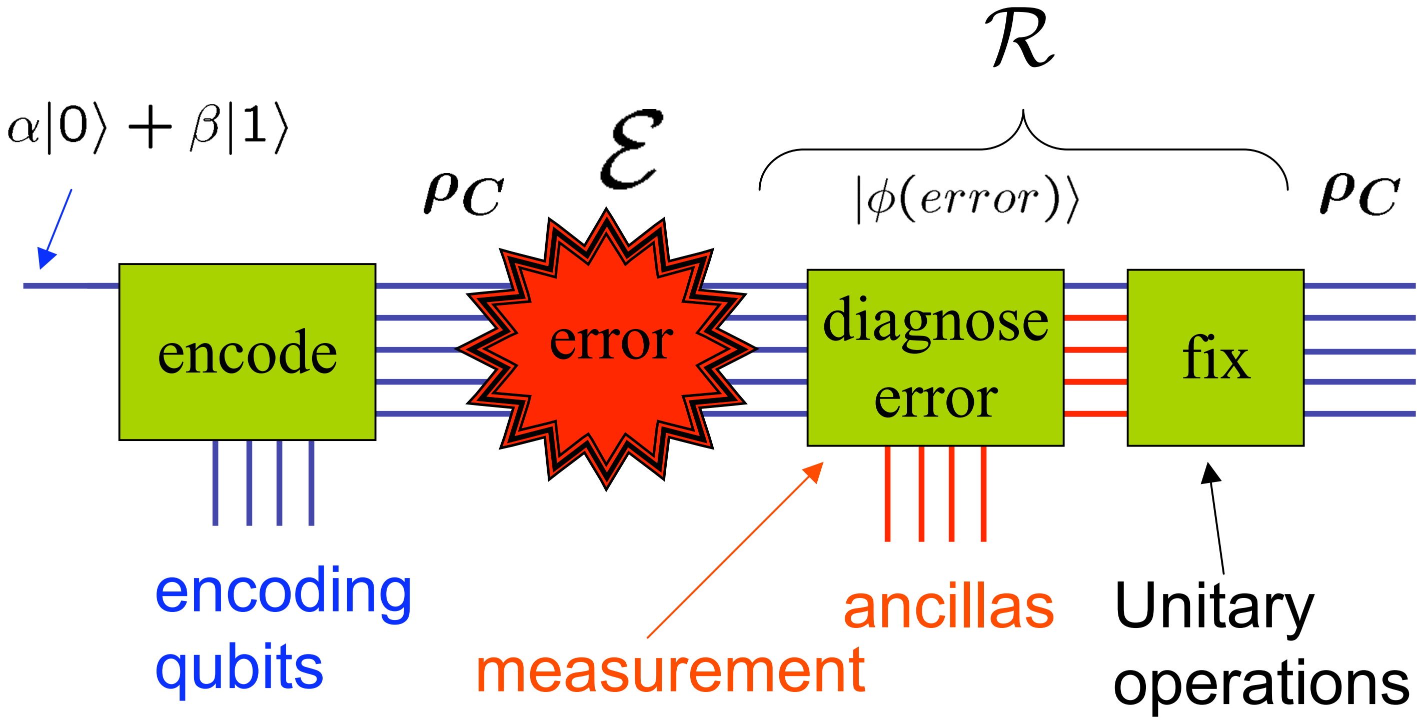
- recovery procedure undertaken

$$\mathcal{R}\left[\varepsilon(\rho_C)\right] = \sum_l R_l \sum_k E_k \rho_C E_k^\dagger R_l^\dagger$$

- regain the encoded state ρ_C

$$\mathcal{R}\left[\varepsilon(\rho_C)\right] = \rho_C$$

Encoding and Recovery



error and recovery are superoperators

$$\rho = \mathbb{S}\left(|\psi\rangle\right) = \sum_k A_k \rho A_k^\dagger$$

Recovery operator $\mathcal{R}$ restores state to the code after error from environment

- encode into a subspace
- no meaurement of state, only of error
- achieve by adding ancilla qubits
- measure ancillas $\rightarrow$ syndrome of error
- perform unitaries conditional on syndrome to correct erroneous qubits

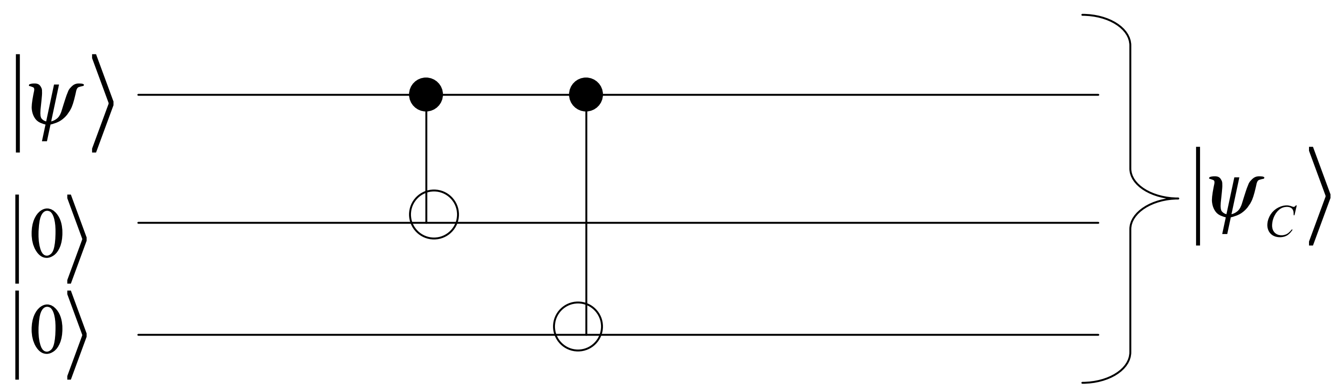
Encoding

e.g., 3-qubit bit flip code

$|0_L>=|000>$

$|1_L>=|111>$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi_C\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$$



$$(\alpha|0\rangle + \beta|1\rangle)\otimes|0\rangle \rightarrow \alpha|00\rangle + \beta|11\rangle$$

$$(\alpha|00\rangle + \beta|11\rangle)\otimes|0\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle \equiv \alpha|0_L\rangle + \beta|1_L\rangle$$

Continuous Errors

$$R_{\theta/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = e^{i\theta/2} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$= \cos(\theta/2) I - i \sin(\theta/2) Z$$

add ancilla(s), transfer error info to ancilla (c-U)

$$Z\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |0_{anc}\rangle \rightarrow Z\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |Z_{anc}\rangle$$

$$I\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |0_{anc}\rangle \rightarrow I\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |no\,error_{anc}\rangle$$

ancilla $\rightarrow$ superposition

$$\cos\left(\frac{\theta}{2}\right) I\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |no\,error_{anc}\rangle$$

$$- i \sin\left(\frac{\theta}{2}\right) Z\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |Z_{anc}\rangle$$

measure ancilla

$$\text{prob. } \sin^2\left(\frac{\theta}{2}\right) \quad Z\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |Z_{anc}\rangle$$

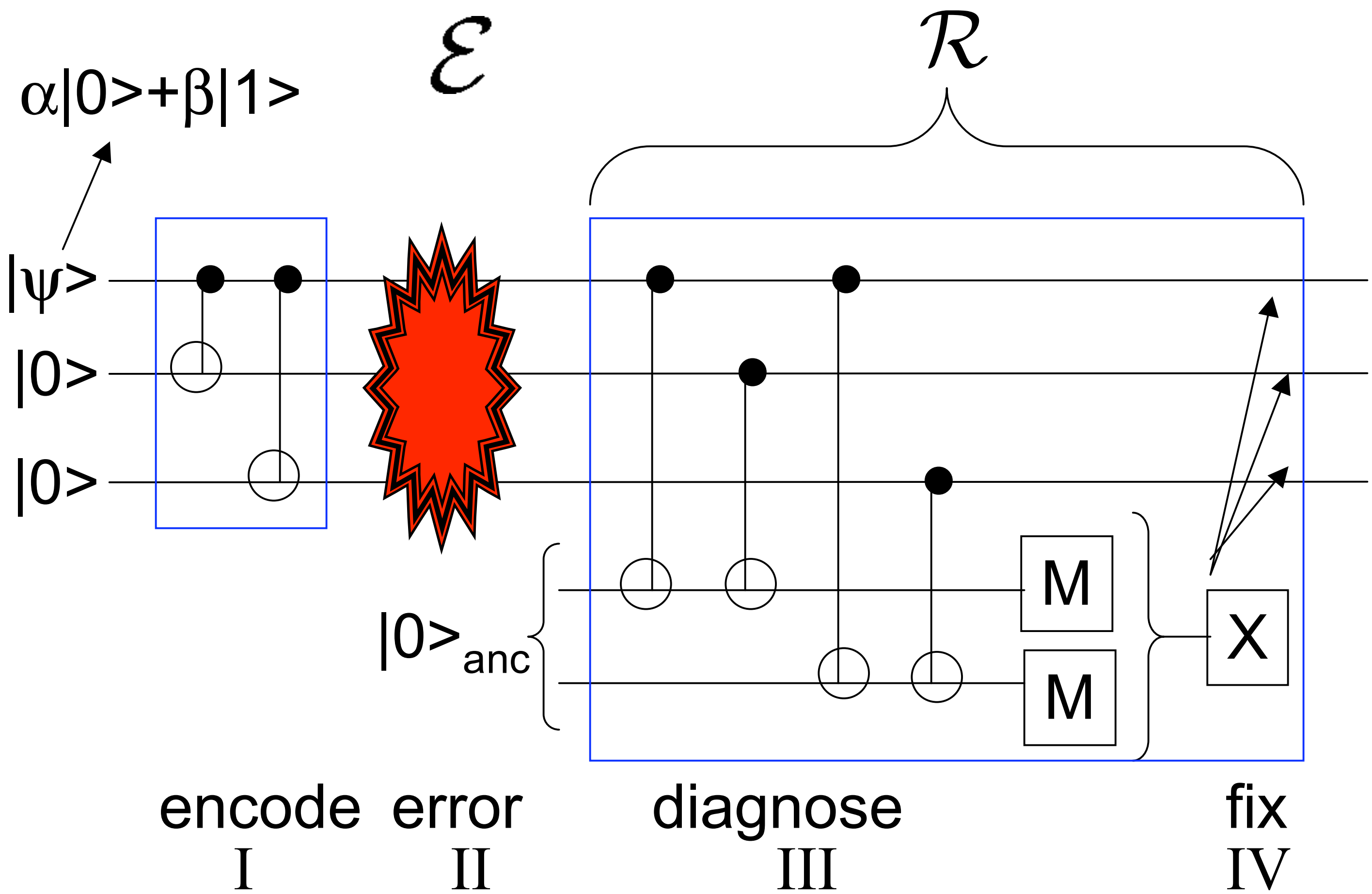
$$\text{prob. } \cos^2\left(\frac{\theta}{2}\right) \quad I\left(\alpha|0_L\rangle + \beta|1_L\rangle\right) \otimes |no\,error_{anc}\rangle$$

invert either one $\rightarrow$ restore initial state

3-qubit Bit Flip Code

$|0_L\rangle = |000\rangle$
$|1_L\rangle = |111\rangle$

Error X with prob. p



I: $(\alpha|0\rangle+\beta|1\rangle)\otimes|0\rangle \otimes|0\rangle \rightarrow \alpha|000\rangle+\beta|111\rangle$

II: 8 possibilities from errors XII, IXI, IIX, XXI, XIX, IXX, XXX, III

state after error	Prob. of getting state
$\alpha\|000\rangle+\beta\|111\rangle$	$(1-p)^3$
$\alpha\|100\rangle+\beta\|011\rangle$	$p(1-p)^2$
$\alpha\|010\rangle+\beta\|101\rangle$	$p(1-p)^2$
$\alpha\|001\rangle+\beta\|110\rangle$	$p(1-p)^2$
$\alpha\|110\rangle+\beta\|001\rangle$	$p^2(1-p)$
$\alpha\|101\rangle+\beta\|010\rangle$	$p^2(1-p)$
$\alpha\|011\rangle+\beta\|100\rangle$	$p^2(1-p)$
$\alpha\|111\rangle+\beta\|000\rangle$	p^3

1 or no error

III: a) perform CNOT between qubits 1 & 2 with ancilla 1
b) perform CNOT between qubits 1 & 3 with ancilla 2

$\rightarrow$

$\alpha\|000\rangle+\beta\|111\rangle\boxed{\|00\rangle}$	$(1-p)^3$
$\alpha\|100\rangle+\beta\|011\rangle\boxed{\|11\rangle}$	$p(1-p)^2$
$\alpha\|010\rangle+\beta\|101\rangle\boxed{\|10\rangle}$	$p(1-p)^2$
$\alpha\|001\rangle+\beta\|110\rangle\boxed{\|01\rangle}$	$p(1-p)^2$
$\alpha\|110\rangle+\beta\|001\rangle\|01\rangle$	$p^2(1-p)$
$\alpha\|101\rangle+\beta\|010\rangle\|10\rangle$	$p^2(1-p)$
$\alpha\|011\rangle+\beta\|100\rangle\|11\rangle$	$p^2(1-p)$
$\alpha\|111\rangle+\beta\|000\rangle\|00\rangle$	p^3

syndrome

syndrome redundant for 1 and 2 (0 and 3) errors, but unequal probabilities

III. c) M = measure ancillas:
assume only 1 (or 0) error $\Rightarrow$ syndrome uniquely identifies error

failure rate of code = rate of $\geq$ 2 errors

$$= 3p^2(1-p)+p^3$$
$$= 3p^2-2p^3$$
$$< p \text{ for } p < 0.5$$

IV. fix by applying unitary conditional on M

syndrome:	00	do nothing
	01	apply σ_x to 3rd qubit
	10	apply σ_x to 2nd qubit
	11	apply σ_x to 1st qubit

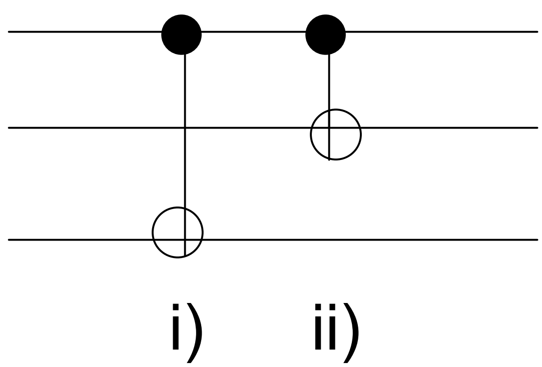
$$\left.\begin{array}{l} \alpha|000\rangle+\beta|111\rangle|00\rangle \\ \alpha|100\rangle+\beta|011\rangle|11\rangle \\ \alpha|010\rangle+\beta|101\rangle|10\rangle \\ \alpha|001\rangle+\beta|110\rangle|01\rangle \end{array}\right\} \text{recover encoded state } \alpha|000\rangle+\beta|111\rangle$$

Decoding

e.g. from syndrome 10

after IV. have $\alpha|000\rangle+\beta|111\rangle$ with $p(1-p)^2$

extract original qubit $\alpha|0\rangle+\beta|1\rangle$ with circuit:



i) $\alpha|000\rangle+\beta|111\rangle \rightarrow \alpha|0\rangle|00\rangle+\beta|1\rangle|10\rangle$

ii) $\alpha|0\rangle|00\rangle+\beta|1\rangle|10\rangle \rightarrow \alpha|0\rangle|00\rangle+\beta|1\rangle|00\rangle$

$= (\alpha|0\rangle+\beta|1\rangle)|00\rangle$

$\Rightarrow$ get correct qubit state with prob. $> 1-p$

prob. of failure $= 3p^2-2p^3 < p$ for $p < 0.5$

success = 100% if no 2 or 3 errors

$\boxed{\text{error prob. reduced from } p \text{ to } O(p^2)}$

3-bit Phase Code

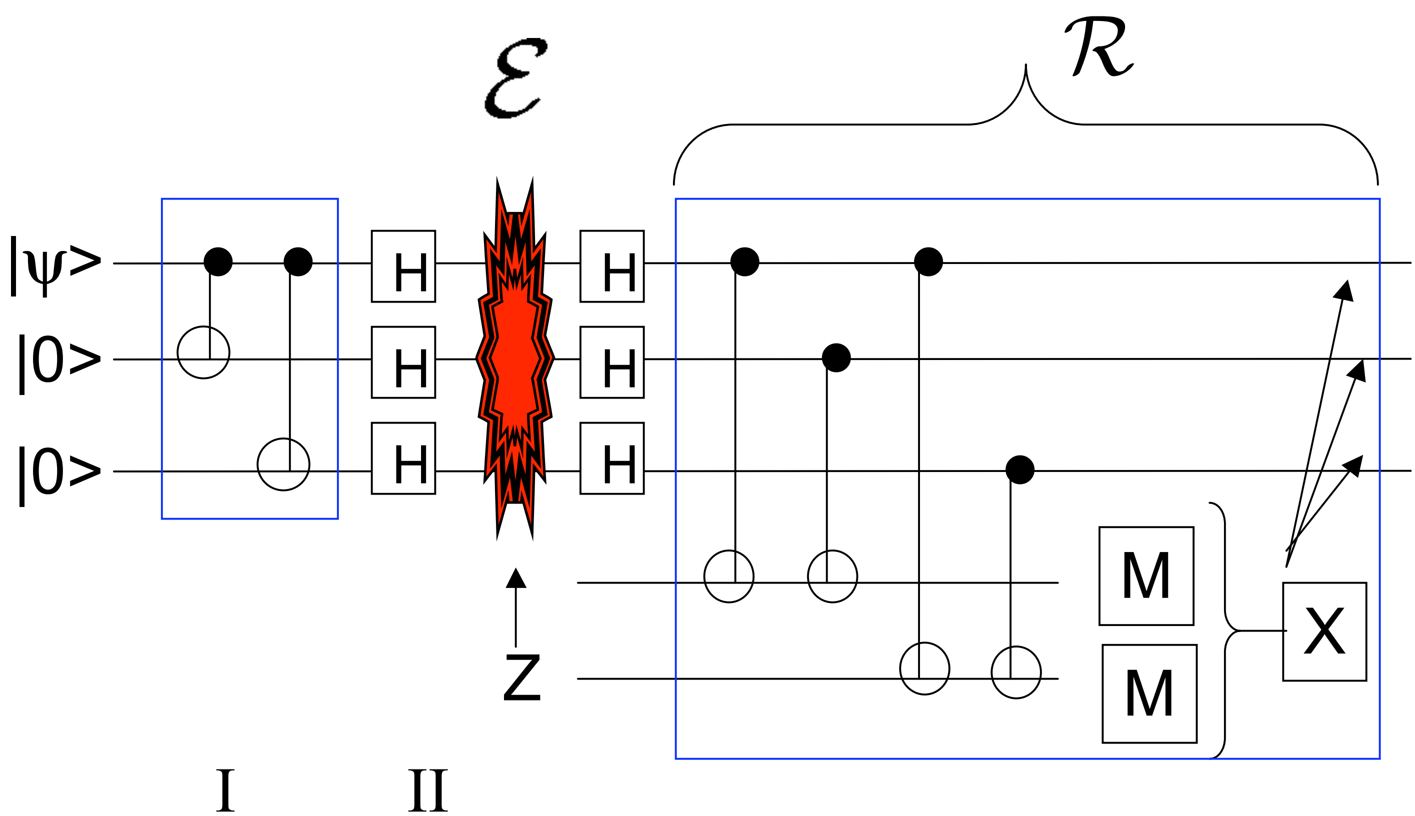
$\sigma_z(\alpha|0\rangle+\beta|1\rangle) = \alpha|0\rangle-\beta|1\rangle$ not classical!

change basis: $|+\rangle = 1/\sqrt{2}(|0\rangle+|1\rangle)$
$|-\rangle = 1/\sqrt{2}(|0\rangle-|1\rangle)$

$$\begin{pmatrix} + \\ - \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = H\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then $\sigma_z|+\rangle=|-\rangle$
$\sigma_z|-\rangle=|+\rangle$
like bit flip!

$H\,\sigma_z H=\sigma_x$ or $H=|+\rangle\langle 0|+|-\rangle\langle 1|$



effectively encoded into $|0_L\rangle = |{+}{+}{+}\rangle$, $|1_L\rangle = |{-}{-}{-}\rangle$

$$I, II \rightarrow \alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$$

phase errors ZII, IZI, ZII act as Z on $|000\rangle$, $|111\rangle$

e.g., $ZII|000\rangle = |000\rangle$

$ZII|111\rangle = -1|111\rangle$

but as X on $|{+}{+}{+}\rangle$, $|{-}{-}{-}\rangle$

Both bit flip and phase errors:

concatenate these two codes:

$|0_L\rangle=(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$
$|1_L\rangle=(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$

inner layer corrects bit flips 000, 111
outer layer corrects phase flips +++, ----

Shor PRA **52**, R2493 (1995)

define Bell basis:

$|000\rangle \pm |111\rangle$
$|001\rangle \pm |110\rangle$
$|010\rangle \pm |101\rangle$
$|100\rangle \pm |011\rangle$

consider decoherence of qubit 1:

$e|0\rangle \rightarrow a_0|0\rangle+a_1|1\rangle$
$e|1\rangle \rightarrow a_2|0\rangle+a_3|1\rangle$

$e, a_0,\ldots a_3 =$ states of env

first triple:

$$|000\rangle+|111\rangle \rightarrow (a_0|0\rangle+a_1|1\rangle)|00\rangle+ (a_2|0\rangle+a_3|1\rangle)|11\rangle$$
$$= a_0|000\rangle+a_1|100|+a_2|011\rangle+a_3|111\rangle$$

put in Bell basis $\rightarrow$

$$=1/2\,(a_0+a_3)\,(|000> + |111>)$$
$$+1/2\,(a_0-a_3)\,(|000> - |111>)$$
$$+1/2\,(a_1+a_2)\,(|100> + |011>)$$
$$+1/2\,(a_1-a_2)\,(|100> - |111>)$$

similarly $|000> - |111>$ goes to

$$=1/2\,(a_0+a_3)\,(|000> - |111>)$$
$$+1/2\,(a_0-a_3)\,(|000> + |111>)$$
$$+1/2\,(a_1+a_2)\,(|100> - |011>)$$
$$+1/2\,(a_1-a_2)\,(|100> + |111>)$$

output 2
(syndrome 2)

assume 1 error only:

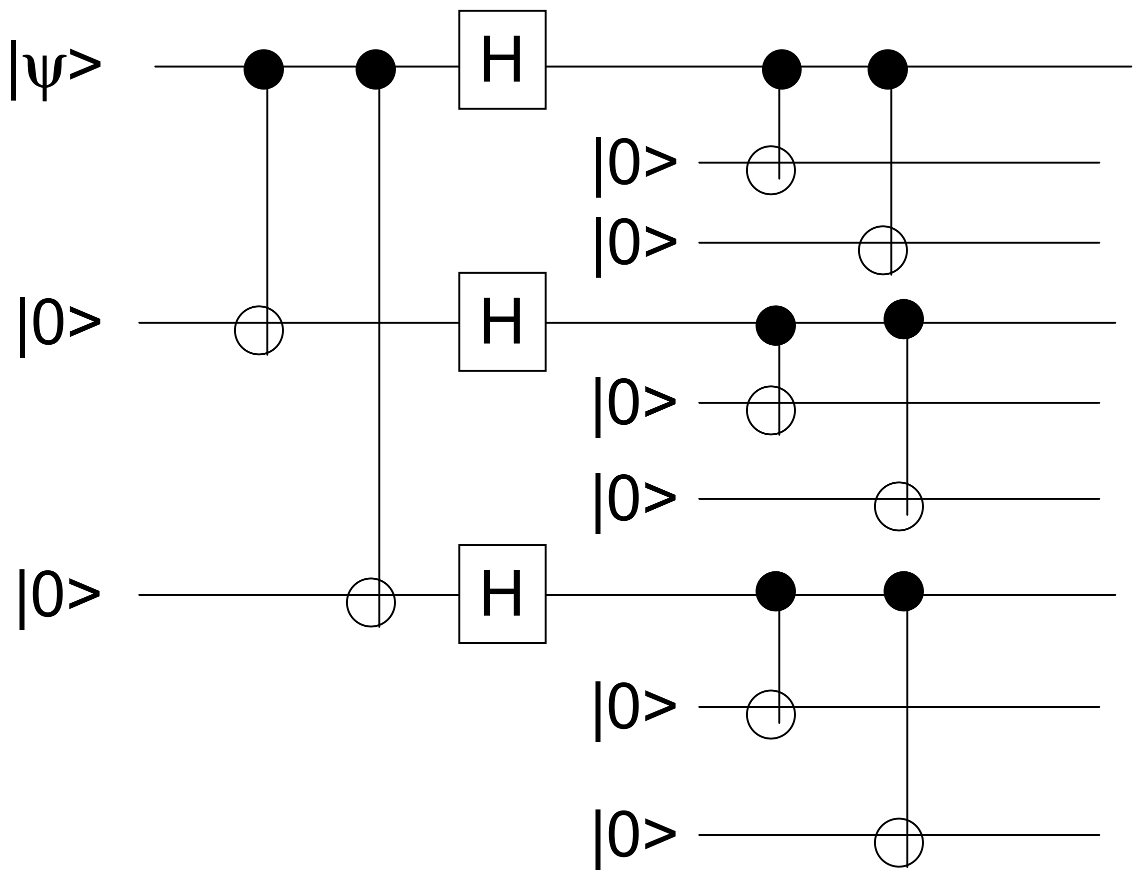
compare all 3 triples, see which differs
majority sign indicates $|0_L>$ or $|1_L>$
find which qubit decohered
(measure 9 ancillas $\rightarrow$ which syndrome)
restore qubit state with a unitary operation

e.g. from $|000> - |111>)$

$1/2\,(a_0+a_3)\,(|000> - |111>) \Rightarrow$ no error
output 2 $1/2\,(a_0-a_3)\,(|000> + |111>) \Rightarrow$ Z error
$1/2\,(a_1+a_2)\,(|100> + |011>) \Rightarrow$ X error
$1/2\,(a_1-a_2)\,(|100> - |011>) \Rightarrow$ ZX=Y error

have diagnosed error on 1st qubit
→ correct with appropriate unitary

Encoder:



[9,1,3] code: 9 physical qubits
1 logical qubit
(3-1)/2=1 arbitrary error corrected

not most efficient code: [7,1,3] and [5,1,3]
cannot compute easily (logical X, Z OK
logical H, CNOT, T hard)