C191 - Lecture 5 - Two-qubit gates, circuit model, teleportation

I. MAKING TWO QUBIT GATES

Suppose we have the Hamiltonian on two qubits,

$$H_1 = J\sigma_z \otimes \sigma_z = J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And single qubit Hamiltonian,

$$H_0 = \frac{1}{2}\gamma B(\sigma_z \otimes I + I \otimes \sigma_z) = \frac{1}{2}\gamma B \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right)$$

We use these Hamiltonians to produce a CPHASE gate:

$$\text{CPHASE} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

By choosing B such that $\frac{1}{2}\gamma B = -J$, we have a total Hamiltonian,

$$H = J \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Now, we can always add an operator proportional to the identity to the Hamiltonian, which simply moves the zero on the energy scale. The effect on the unitary operator generated by the Hamiltonian is to multiply it by a complex phase - check this! So adding $JI \otimes I$ gives us

$$H' = J \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

A system evolving under this Hamiltonian for a time t experiences a unitary operator,

$$U = \exp(-iH't/\hbar) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-4iJt/\hbar} \end{pmatrix}$$

Choosing t such that $4Jt/\hbar = \pi$, we generate our CPHASE gate.

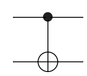
II. QUANTUM CIRCUITS

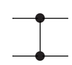
Important quantum circuits and their matrix representations

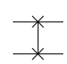
A. Single qubit operators

Pauli X	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$-\boxed{X}-$
Pauli Y	$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$-\boxed{Y}-$
Pauli Z	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$-\boxed{Z}-$
Hadamard	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$-\boxed{H}-$
S	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$-\boxed{S}-$
T	$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$	$-\boxed{T}-$

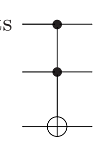
B. Two qubit operators

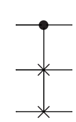
CNOT $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ 

CPHASE $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ 

SWAP $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 

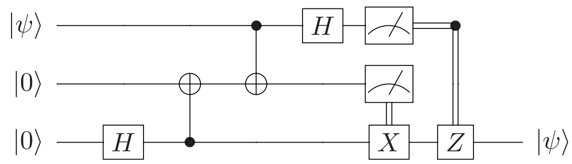
C. Three qubit operators

Toffoli (Controlled-CNOT) If first qubit is $|1\rangle$, perform a CNOT on the second and third qubits 

Fredkin (Controlled-SWAP) If first qubit is $|1\rangle$, swap the second and third qubits 

III. TELEPORTATION

Let's work out the details of the following quantum circuit:



The initial state is

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle$$

After the first Hadamard, the state becomes:

$$|\psi\rangle \otimes |0\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

The CNOT puts the last qubits into a maximally entangled state:

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$$

Assuming the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, the state at this point is,

$$(\alpha\,|0\rangle + \beta\,|1\rangle) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$$

We'll expand this out and drop the tensor product signs:

$$\frac{1}{\sqrt{2}} \left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\right)$$

After the second CNOT, the state becomes:

$$\frac{1}{\sqrt{2}} \left(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle\right)$$

And after the Hadamard, the state becomes,

$$\frac{1}{2} \left(\alpha\,|000\rangle + \alpha\,|100\rangle + \alpha\,|011\rangle + \alpha\,|111\rangle + \beta\,|010\rangle - \beta\,|110\rangle + \beta\,|001\rangle - \beta\,|101\rangle\right)$$

The measurement results, the resulting state of the third qubit, the correction operator, and the resulting state are summarized in the following table: So this circuit uses two bits of classical information and one shared Bell pair to

Qubit 1	Qubit 2	State of qubit 3	Correction step	Final state
0	0	$\alpha\,|0\rangle + \beta\,|1\rangle$		$\alpha\,|0\rangle + \beta\,|1\rangle$
0	1	$\beta\,|0\rangle + \alpha\,|1\rangle$	X	$\alpha\,|0\rangle + \beta\,|1\rangle$
1	0	$\alpha\,|0\rangle - \beta\,|1\rangle$	Z	$\alpha\,|0\rangle + \beta\,|1\rangle$
1	1	$-\beta\,|0\rangle + \alpha\,|1\rangle$	ZX	$\alpha\,|0\rangle + \beta\,|1\rangle$

move a state from qubit 1 to qubit 3. Neat!