C191 - Lectures 8 and 9 - Measurement in Quantum Mechanics

I. THE MEASUREMENT POSTULATE

We've discussed before that the probability of measuring a given state is equal to the squared modulus of the amplitude associated with that state. This is known as *Bohr's rule*. So for instance, if

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

then the probability of measuring the system to be in state $|0\rangle$ is $|\alpha|^2$, while the probability of measuring the state $|1\rangle$ is $|\beta|^2$. If we wished instead to measuring in a different basis, this simple rule would first require us to express our state in terms of that basis. So a measurement in the $|\pm\rangle$ basis would require the state to be rewritten as

$$|\psi\rangle = \frac{\alpha + \beta}{\sqrt{2}} |+\rangle + \frac{\alpha - \beta}{\sqrt{2}} |-\rangle$$

From this, we can now read off the probabilities: $|\alpha + \beta|^2 /2$ that we see $|+\rangle$ and $|\alpha - \beta|^2 /2$ that we see $|-\rangle$. But this approach can get a little clumsy, so we instead will define a more rigorous mathematical formalism describing measurement that will be applicable to a much broader range of measurement scenarios (such as measuring only one of several qubits in a system, or measurements which occasionally fail). We shall begin by describing the formalism, then we'll work with it by practicing some examples.

A measurement in quantum mechanics consists of a set of measurement operators $\{M_m\}_{m=1}^n$. The index m refers to the measurement outcome. Assume the state of the system immediately preceding the measurement is $|\psi\rangle$.

1. The probability of observing measurement outcome m is $\langle\psi|M_m^\dagger M_m|\psi\rangle$.

2. Because the total probability over all measurement outcomes must sum to 1, the above implies that

$$\sum_m M_m^\dagger M_m = \mathcal{I}$$

   Where $\mathcal{I}$ is the identity operator. This is known as the *completeness relation*.

3. Assuming outcome m has been observed, the state of the system immediately following the measurement is

$$|\psi\rangle \to \frac{M_m |\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

   This is often called "wave function collapse," as measurement appears to make a complicated quantum state *collapse* into a state consistent with the measurement.

This description of measurement in terms of the measurement operators should be taken as a postulate of quantum mechanics. That is, it cannot be *derived*, but instead *defines* the mathematical structure of quantum mechanics. Now let's see how the formalism bears out in practice.

II. PROJECTIVE MEASUREMENTS

For now, we shall focus entirely on *projective measurements*. These measurements are the simplest type which appear in quantum mechanics and can be thought of as ideal, perfect measurements of a quantum system [1]. The measurement operators of projective measurements are *projectors*, operators P which satisfy $P^2 = P$. An example of a projection operator is the operator $|0\rangle\langle 0|$. We can see that it is, in fact a projector,

$$(|0\rangle\langle 0|)^2 = |0\rangle\langle 0| \, |0\rangle\langle 0| = |0\rangle \left(\langle 0|0\rangle\right) \langle 0| = |0\rangle\langle 0|$$

[1] Note that this is in contrast to the most general form, the *postive, operator-valued measurement*, which can be used to describe, among other things, imperfect measurements which occasionally fail. We'll discuss these later in the course.

A measurement in the $|0\rangle, |1\rangle$ basis would then correspond to projectors onto each of the two states,

$$M_0 = |0\rangle\langle 0| \quad M_1 = |1\rangle\langle 1|$$

We should check that these two operators satisfy the completeness relation:

$$M_0^\dagger M_0 + M_1^\dagger M_1 = |0\rangle\langle 0| \, |0\rangle\langle 0| + |1\rangle\langle 1| \, |1\rangle\langle 1| = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathcal{I}$$

So we're good. Assume our system is in the state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle .$$

We can now use the axioms to determine the probability of measuring the system in state $|0\rangle$ to be:

$$\begin{aligned}\left\langle \psi | M_0^\dagger M_0 | \psi \right\rangle &= \langle \psi | 0 \rangle \langle 0 | \psi \rangle \\ &= \left(\alpha^* \langle 0| + \beta^* \langle 1|\right) \left(|0\rangle\langle 0|\right) \left(\alpha |0\rangle + \beta |1\rangle\right) \\ &= |\alpha|^2\end{aligned}$$

And the state after measuring the system in $|0\rangle$ is, as expected,

$$|\psi\rangle \to M_0 |\psi\rangle \sqrt{\left\langle \psi | M_0^\dagger M_0 | \psi \right\rangle} = \frac{1}{|\alpha|} |0\rangle\langle 0| \left(\alpha |0\rangle + \beta |1\rangle\right) = \frac{\alpha}{|\alpha|} |0\rangle$$

But note that $\alpha = |\alpha| e^{i\phi}$ for some phase angle $\phi \in [0, 2\pi)$. This means that $\alpha/|\alpha| = e^{i\phi}$ is a complex phase. Recalling that global phases don't matter, we can therefore say that the state after measurement is simply $|0\rangle$.

We can repeat the calculation for measurement in the $|\pm\rangle$ basis, for which the measurement operators are,

$$M_+ = |+\rangle\langle +| = \frac{1}{2} \left(|0\rangle + |1\rangle\right) \left(\langle 0| + \langle 1|\right)$$

$$M_- = |-\rangle\langle -| = \frac{1}{2} \left(|0\rangle - |1\rangle\right) \left(\langle 0| - \langle 1|\right)$$

After working through the algebra, you will see that the predictions agree with our discussion above.

III. MULTIPLE QUBITS

The formalism extends quite naturally to multiple qubits. Let's consider a two-qubit state,

$$|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle$$

One example of a complete set of measurement operators would be

$$M_{00} = |00\rangle\langle 00| \quad M_{01} = |01\rangle\langle 01| \quad M_{10} = |10\rangle\langle 10| \quad M_{11} = |11\rangle\langle 11|$$

However, it is often the case that our systems are quite large and we can only measure one qubit at a time. If we were to only measure the first qubit, we would be unable to distinguish, for example, the states $|00\rangle$ and $|01\rangle$. We must then construct new measurement operators which are also unable to make this distinction. These operators will also be projectors, but will project onto all states consistent with the measurement. So if we measure the first qubit to be in the state $|0\rangle$, the corresponding measurement operator will consist of projectors onto the states $|00\rangle$ and $|01\rangle$:

$$M_0^{(1)} = M_{00} + M_{01} \quad = |00\rangle\langle 00| + |01\rangle\langle 01| = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \quad = |0\rangle\langle 0| \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = |0\rangle\langle 0| \otimes \mathcal{I}$$

Similarly,

$$M_1^{(1)} = M_{10} + M_{11} \quad = |10\rangle\langle 10| + |11\rangle\langle 11| = |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \quad = |1\rangle\langle 1| \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = |1\rangle\langle 1| \otimes \mathcal{I}$$

Where we have used the notation $M_i^{(j)}$ to mean the measurement operator corresponding to measuring i on the j^{th} qubit. As an exercise, you should verify that these operators are projectors. The probability of measuring the first qubit in state $|0\rangle$ is therefore,

$$\begin{aligned}\text{Prob} &= \left\langle \psi | M_0^\dagger M_0^{(1)} | \psi \right\rangle \\ &= \left(\alpha^* \langle 00| + \beta^* \langle 01| + \gamma^* \langle 10| + \delta^* \langle 11|\right) \left(|0\rangle\langle 0| \otimes \mathcal{I}\right) \left(\alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle\right) \\ &= \left(\alpha^* \langle 00| + \beta^* \langle 01|\right) \left(\alpha |00\rangle + \beta |01\rangle\right) \\ &= |\alpha|^2 + |\beta|^2\end{aligned}$$

Which is the sum of the probabilities of measuring $|00\rangle$ and $|01\rangle$, as expected. The state after measurement is then

$$\begin{aligned}|\psi\rangle \rightarrow &\frac{1}{\sqrt{\text{Prob}}} M_0^{(1)} |\psi\rangle \\ = &\frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \left(|0\rangle\langle 0| \otimes \mathcal{I}\right) \left(\alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle\right) \\ = &\frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \left(\alpha |00\rangle + \beta |01\rangle\right)\end{aligned}$$

So the relative probability of observing $|00\rangle$ and $|01\rangle$ hasn't changed, but by measuring $|0\rangle$ on the first qubit, we have eliminated the possibility that the system is in either state $|10\rangle$ or $|11\rangle$.

IV. PHYSICAL MEASUREMENT

To actually perform a measurement, we bring our system into contact with a meter. The meter, since it is just another physical object, is a quantum system whose initial state we shall suppose is blank, which we'll notate as $|?\rangle$. When the system and the meter are brought into contact, they interact with one another. The experimentalist has chosen the meter and the coupling mechanism so that the evolution of the meter's state is conditional on the state of the system. For a single qubit measured in the $0, 1$ basis, this means that the meter will evolve to the state $|\mu_0\rangle$ if the system is in the state $|0\rangle$ and will evolve to $|\mu_1\rangle$ if the system is in the state $|1\rangle$. The unitary operator describing this process is then,

$$U_{\text{meas}} = |0\rangle\langle 0| \otimes \left(|\mu_0\rangle\langle ?| + |?\rangle\langle \mu_0|\right) + |1\rangle\langle 1| \otimes \left(|\mu_1\rangle\langle ?| + |?\rangle\langle \mu_1|\right)$$

Check that this is unitary! Note that for projective measurements, the system operators are exactly the measurement operators corresponding to measurement in the $|0, 1\rangle$ basis. If the system is initially in the state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, then after the meter has been brought into contact with the system, their combined state is

$$U_{\text{meas}} |\psi\rangle |?\rangle = \alpha |0\rangle |\mu_0\rangle + \beta |1\rangle |\mu_1\rangle$$

So the system is now *entangled* with the meter. Let's now discard the qubit, since we are only interested in the state of the meter. When we discard part of a quantum system, generically we are no longer left with a pure quantum state, but rather a density matrix[2].

To see what happens when we "throw away" a component of a quantum system, let's consider a two qubit system in the state $(|00\rangle + |11\rangle)/\sqrt{2}$. We now stop looking at the second qubit and assume we will never be able to look at it again. We might give it to someone else (Alice) who then takes it light-years away from us, for example. Anything that happens to that qubit is then completely unable to affect the state of the qubit we kept. For all we know, Alice could have measured it. Because our state is entangled, we know that any measurement results must be correlated, so if Alice measures her qubit in state $|0/1\rangle$, then she knows our qubit must also be in state $|0/1\rangle$. Alice has a 50/50 chance of measuring 0 or 1, so, that tells us that our qubit is in state $|0\rangle$ with probability 1/2 or in state $|1\rangle$ with probability 1/2. The mathematical object we use to describe quantum systems where there is uncertainty in the

[2] Later we will discuss the *partial trace*, the rigorous way to eliminate quantum systems, but for now we will just make a few motivating arguments.

quantum state is the density matrix. If the quantum system is in state $|\psi_i\rangle$ with probability p_i, then the density matrix is defined as,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

So after we get rid of our second qubit, the qubit we have left can be described by the density matrix,

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

So what does this have to do with our discussion about the meter? Well, after throwing out the qubit, we have a meter whose state is best described by a density operator,

$$\rho_{\text{meter}} = |\alpha|^2 |\mu_0\rangle\langle\mu_0| + |\beta|^2 |\mu_1\rangle\langle\mu_1|$$

This state represents a *classical* probability distribution over the possible meter configurations, where the probabilities are given as $|\alpha|^2$ to be in state $|\mu_0\rangle$ and $|\beta|^2$ to be in state $|\mu_1\rangle$. Unfortunately, that's the best we can do to describe the physics of a measurement: a quantum system in a superposition state is combined with a meter, leaving the two in an entangled state - then the qubit is thrown out, and the state of the meter is a classical probability distribution. Quantum mechanics utterly fails at telling us how wave function collapse actually works. We could keep adding meters, building ourselves up to a complete quantum description of the observer (you!) and the universe, but as they say, it's turtles all the way down.

A. Repeated measurements

One of the advantages of this approach to measurement is that it shows very clearly how two measurements can fail to commute with one another. Suppose we had a qubit in the usual generic state,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

And we have *two* meters. One of which corresponds to a measurement in the 0,1 basis and couples to the system according to the operator,

$$U_{0/1} = |0\rangle\langle 0| \otimes (|\mu_0\rangle\langle ?| + |?\rangle\langle\mu_0|) + |1\rangle\langle 1| \otimes (|\mu_1\rangle\langle ?| + |?\rangle\langle\mu_1|)\,,$$

while the other corresponds to a measurement in the $\pm$ basis and couples to the system according to the operator,

$$U_{+/-} = |+\rangle\langle +| \otimes (|\mu_+\rangle\langle ?| + |?\rangle\langle\mu_+|) + |-\rangle\langle -| \otimes (|\mu_-\rangle\langle ?| + |?\rangle\langle\mu_-|)\,.$$

Let's perform a measurement in the 0,1 basis first, then the plus/minus basis. After the first meter is attached and allowed to evolve, the combined state is

$$\alpha|0\rangle|\mu_0\rangle + \beta|1\rangle|\mu_1\rangle$$

Combing the second meter, we should first write the state in the plus-minus basis,

$$\frac{\alpha}{\sqrt{2}}|+\rangle|\mu_0\rangle|?\rangle + \frac{\alpha}{\sqrt{2}}|-\rangle|\mu_0\rangle|?\rangle + \frac{\beta}{\sqrt{2}}|+\rangle|\mu_1\rangle|?\rangle - \frac{\beta}{\sqrt{2}}|-\rangle|\mu_1\rangle|?\rangle$$

Then allowing the system to evolve,

$$\frac{\alpha}{\sqrt{2}}|+\rangle|\mu_0\rangle|\mu_+\rangle + \frac{\alpha}{\sqrt{2}}|-\rangle|\mu_0\rangle|\mu_-\rangle + \frac{\beta}{\sqrt{2}}|+\rangle|\mu_1\rangle|\mu_+\rangle - \frac{\beta}{\sqrt{2}}|-\rangle|\mu_1\rangle|\mu_-\rangle$$

The probabilities of the various measurement outcomes are then,

outcome	probability
0, +	$\lvert\alpha\rvert^2/2$
0, -	$\lvert\alpha\rvert^2/2$
1, +	$\lvert\beta\rvert^2/2$
1, -	$\lvert\beta\rvert^2/2$

Repeating the calculation, but now performing the first measurement in the plus-minus basis, we first have the state

$$\frac{\alpha}{\sqrt{2}}|+\rangle|?\rangle+\frac{\alpha}{\sqrt{2}}|-\rangle|?\rangle+\frac{\beta}{\sqrt{2}}|+\rangle|?\rangle-\frac{\beta}{\sqrt{2}}|-\rangle|?\rangle$$

Evolving under $U_{+/-}$, we have

$$\frac{\alpha}{\sqrt{2}}|+\rangle|\mu_+\rangle+\frac{\alpha}{\sqrt{2}}|-\rangle|\mu_-\rangle+\frac{\beta}{\sqrt{2}}|+\rangle|\mu_+\rangle-\frac{\beta}{\sqrt{2}}|-\rangle|\mu_-\rangle$$

Rewriting in terms the 0,1 basis, we have

$$\frac{\alpha+\beta}{2}|0\rangle|\mu_+\rangle+\frac{\alpha+\beta}{2}|0\rangle|\mu_-\rangle+\frac{\alpha-\beta}{2}|1\rangle|\mu_+\rangle+\frac{\alpha-\beta}{2}|1\rangle|\mu_-\rangle$$

Coupling to a 0,1 meter and allowing the state to evolve, we have

$$\frac{\alpha+\beta}{2}|0\rangle|\mu_+\rangle|\mu_0\rangle+\frac{\alpha+\beta}{2}|0\rangle|\mu_-\rangle|\mu_0\rangle+\frac{\alpha-\beta}{2}|1\rangle|\mu_+\rangle|\mu_1\rangle+\frac{\alpha-\beta}{2}|1\rangle|\mu_-\rangle|\mu_1\rangle$$

The probabilities of the various measurement outcomes are then,

outcome	probability
0, +	$\left\lvert\frac{\alpha+\beta}{2}\right\rvert^2/2$
0, -	$\left\lvert\frac{\alpha+\beta}{2}\right\rvert^2/2$
1, +	$\left\lvert\frac{\alpha-\beta}{2}\right\rvert^2/2$
1, -	$\left\lvert\frac{\alpha-\beta}{2}\right\rvert^2/2$

It is important to notice is that the measurement results are different depending on the order we perform the measurements. This calculation can also be done directly using measurement operators.

V. HEISENBERG'S UNCERTAINTY PRINCIPLE

The incompatibility of various measurements can be quantified by *Heisenberg's uncertainty principle*. Suppose we have two Hermitian operators, A, B and a state $|\psi\rangle$. Then the expectation value of the product AB is,

$$\langle AB\rangle=\langle\psi|AB|\psi\rangle=x+iy$$

for some real numbers x, y. Taking the hermitian conjugate of the above equation, we get

$$\langle BA\rangle=\langle\psi|BA|\psi\rangle=x-iy$$

Using these equations, we can construct the expectation values of the commutator, $[A,B]=AB-BA$, and anticommutator, $\{A,B\}=AB+BA$:

$$\langle[A,B]\rangle=2iy$$

$$\langle\{A,B\}\rangle=2x$$

Squaring these and adding them together we get,

$$|\langle[A,B]\rangle|^2+|\langle\{A,B\}\rangle|^2=4(x^2+y^2)$$

But squaring the expectation of AB, we get,

$$|\langle AB\rangle|^2=x^2+y^2$$

Combining these, we have

$$|\langle [A,B]\rangle|^2 + |\langle \{A,B\}\rangle|^2 = 4\,|\langle AB\rangle|^2$$

We now use the Cauchy-Schwartz inequality, which states that

$$|\langle AB\rangle|^2 \leq \left\langle A^2\right\rangle \left\langle B^2\right\rangle.$$

Leaving us with

$$|\langle [A,B]\rangle|^2 + |\langle \{A,B\}\rangle|^2 \leq 4\left\langle A^2\right\rangle \left\langle B^2\right\rangle$$

The second term on the LHS is positive, so we can drop it without invalidating the inequality. Then taking the positive square-root leaves us with

$$\frac{1}{2}\,|\langle [A,B]\rangle| \leq \sqrt{\langle A^2\rangle \langle B^2\rangle}.$$

Now notice that we can add or subtract a constant to both A and B without affecting the LHS. By replacing,

$$A \to A - \langle A\rangle \quad B \to B - \langle B\rangle$$

the RHS contains terms like

$$\sqrt{\langle (A-\langle A\rangle)^2\rangle}$$

But this is simply the variance of A, denoted ΔA. Our inequality then becomes

$$\frac{1}{2}\,|\langle [A,B]\rangle| \leq \Delta A\,\Delta B.$$

Let's see how this inequality comes into practice for spin systems. Let's let $A=\sigma_x$, $B=\sigma_y$, and $|\psi\rangle = |0\rangle$. The commutator $[\sigma_x,\sigma_y] = 2i\sigma_z$, so we have

$$\frac{1}{2}\,|\langle 0|\,2i\sigma_z\,|0\rangle| = 1 \leq \Delta\sigma_x\,\Delta\sigma_z.$$

Which means that the variance of both σ_x and σ_y must be strictly greater than 0, implying that we cannot know both with perfect accuracy. If two operators commute, however, then it is possible to have a state for which both both operators have zero expected variance.