Writing Tail-Recursive Functions

Tail-recursive functions directly return the value of their recursive call. This worksheet will present a way of writing and re-writing recursive functions so that they are tail-recursive. Examples will be given in both Scheme and Python in order to aid understanding, however you should keep in mind that only Scheme has tail-recursion capabilities.

Difficulties in Writing Tail-Recursive Functions

In a tail-recursive function, we cannot modify the value of the recursive call before returning it. We must return it directly.

```scheme
(define (sum lst)
   ; Sums a list of numbers.
   (if (null? lst)
      0
      (+
         (car lst)
         (sum (cdr lst))
      )
   )
)
```

```python
def sum(lst):
   """Sums a list of numbers."""
   if len(lst) == 0:
      return 0
   return lst[0] + sum(lst[1:])
```

This function is not tail-recursive because after getting the value of the recursive call, it adds the first element of the list to it.

```scheme
(define (sum lst)
   (if _____
      _____
      (sum _____)
   )
)
```

```python
def sum(lst):
   if _____:
      return _____
   return sum(_____)
```

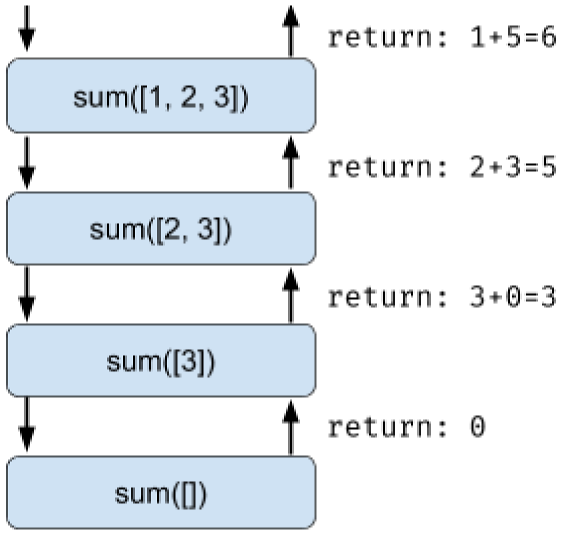
This is what the function would look like if it were tail-recursive. But how to write it in this format?

Pass-Up and Pass-Down Recursion

In order to see how to write `sum` in a tail-recursive way, we will take a detour to look at two general styles of writing recursive functions, “pass-up” and “pass-down” style. Pass-up style is often more natural and obvious, but it is pass-down style that enables tail-recursion.

```
(define (sum lst)
   ; Sums a list of numbers.
   (if (null? lst)
      0
      (+
         (car lst)
         (sum (cdr lst))
      )
   )
)
```

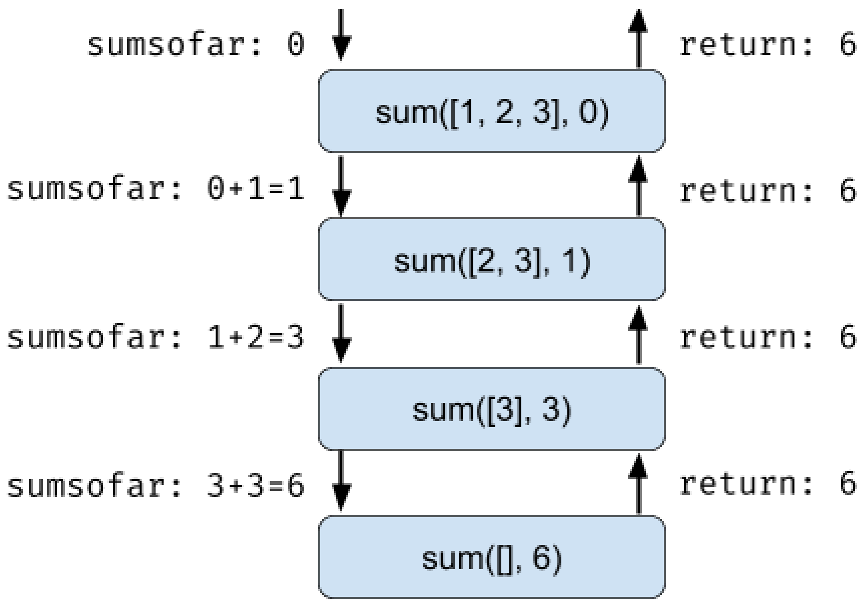
```
def sum(lst):
   """Sums a list of numbers."""
   if len(lst) == 0:
      return 0
   return lst[0] + sum(lst[1:])
```



The version of `sum` we saw above was written in pass-up style. In pass-up recursion, partial solutions flow up the recursive call chain, through the return values. No useful computation occurs on the way down the recursive call chain. `sum([1, 2, 3])` immediately calls `sum([2, 3])` without doing anything; `sum([2, 3])` then calls `sum([3])`, and so on. The useful computation occurs as we return back up the recursive call chain. `sum([])` returns 0 to `sum([3])`. `sum([3])` takes that `0` and returns `3 + 0 = 3` to `sum([2, 3])`. `sum([2, 3])` takes that `3` and so on.

```scheme
(define (sum lst sumsofar)
   ; Sums a list of numbers.
   (if (null? lst)
      sumsofar
      (sum
         (cdr lst)
         (+ (car lst) sumsofar)
      )
   )
)
```

```python
def sum(lst, sumsofar):
   """Sums a list of numbers."""
   if len(lst) == 0:
      return sumsofar
   newsum = lst[0] + sumsofar
   return sum(lst[1:], newsum)
```



This version of `sum` is written in pass-down style. In pass-down recursion, partial solutions flow down the recursive call chain, through the arguments. All the useful computation occurs on the way down the recursive call chain; the return values simply pass the final answer all the way back up. `sum([1, 2, 3], 0)` does `1+0=1` and returns `sum([2, 3], 1)`. `sum([2, 3], 1)` does `2+1=3` and returns `sum([3], 3)`.

Pass-Down Recursion Enables Tail-Recursiveness

In pass-down recursion, the value of the recursive call is returned directly, which makes it possible to write a tail-recursive function. The pass-down version of `sum` above is tail-recursive.

Pass-Down Recursion May Require Adding an Argument

To write a recursive function in the pass-down style, you may need to add an argument that represents the "result so far". If writing a function that sums a list of numbers, the “result so far” argument will represent the sum of the *previous* numbers in the list. If writing a function that finds the maximum of a list of numbers, the “result so far” argument will represent the maximum of the *previous* numbers in the list.

Pass-Down Recursion May Require Using a Helper Function

Our added “result so far” argument is a bit annoying for people who actually use our function; they always must pass an initial value for that additional argument. For example, notice how when we rewrote sum in pass-down style, it had to be called as `(sum ‘(1 2 3) 0)` instead of `(sum ‘(1 2 3))`. We can remove this annoying additional argument using a helper function. We will add the additional argument to the helper function instead of the original function. The original function’s job will simply be to call the helper function.

```
(define (sum lst)
   ; Sums a list of numbers.
   (define (helper lst sumsofar)
      (if (null? lst)
         sumsofar
         (sum
            (cdr lst)
            (+ (car lst) sumsofar)
         )
      )
   )
   (helper lst 0)
)
```

```
def sum(lst):
   """Sums a list of numbers."""
   def helper(lst, sumsofar):
      if len(lst) == 0:
         return sumsofar
      newsum = lst[0] + sumsofar
      return sum(lst[1:], newsum)
   return helper(lst, 0)
```

Practice Problems

Reverse List

Write a tail-recursive function that reverses `lst`.

```
(define (reverse lst)
   (define (f l r)
      (if __________
         __________
         __________
      )
   )
   __________
)
```

Raise Number to Power

Write a tail-recursive function that raises b to the n-th power using multiplication.

```
(define (power b n)
   (define (f p k)
      (if __________
         __________
         __________
      )
   )
   __________
)
```

Hailstone

Write a tail-recursive function that finds the length of the hailstone sequence that starts with n.

```
(define (hailstone n)
   (define (f k i)
      (if __________
         __________
         __________
         __________
         __________
      )
   )
   __________
)
```

Solutions

Reverse List

```
(define (reverse lst)
      (define (f l r)
            (if (null? l)
                  r
                  (f (cdr l) (cons (car l) r))
            )
      )
      (f lst nil)
)
```

Raise Number to Power

```
(define (power b n)
      (define (f p k)
            (if (= p 0)
                  k
                  (f (- p 1) (* k b))
            )
      )
      (f n 1)
)
```

Hailstone

```
(define (hailstone n)
      (define (f k i)
            (if (= k 1)
                  i
                  (if (= (modulo k 2) 0)
                        (f (/ k 2) (+ i 1))
                        (f (+ (* k 3) 1) (+ i 1))
                  )
            )
      )
      (f n 1)
)
```