CS 61A/CS 98-52

Mehrdad Niknami

University of California, Berkeley

Warning

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Ready?!

Preliminaries

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- "Given F and initial guess x_0, find x that minimizes $F(x)$"

Brachistochrone Problem

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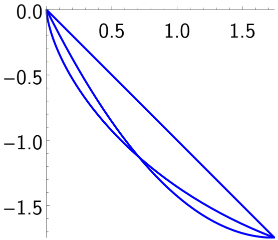

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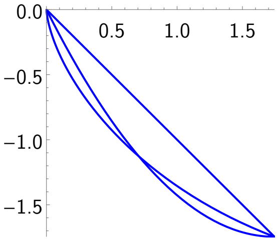

$\Longrightarrow$ How would **you** solve this?

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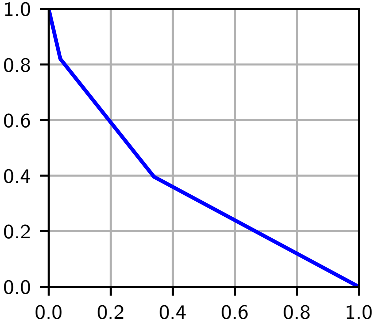

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  - Good luck in N-D... infinitely many directions

Algorithm

Newton-Raphson method for **optimization**:

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f(x_k) - f(x_{k+1}) &= f'(x_k)\,(x_k - x_{k+1}) \\
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4. ...Profit?

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2. Newton coordinate-descent

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Doesn't work as often, but it works very well here.

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    delta = 0.0
    for i in range(1, len(coords) - 1):
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            def  f(c): return derivative(F, c, i, j, h)
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def derivative(f, coords, i, j, h):
    x = coords[i][j]
    coords[i][j] = x + h;  f2 = f(coords)
    coords[i][j] = x - h;  f1 = f(coords)
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    return (f2 - f1) / (2 * h)
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∼ Words of Wisdom ∼

If your problem has {`fundamental feature`} that your solution doesn't, you've created more problems.

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```
def falling_time(coords): # coords = [[x1,y1], [x2,y2], ...]
    t, speed = 0.0, 0.0
    prev = None
    for coord in coords:
        if prev != None:
            dy = coord[1] - prev[1]
            d = ((coord[0] - prev[0]) ** 2 + dy ** 2) ** 0.5
            accel = -9.80665 * dy / d
            for dt in quadratic_roots(accel, speed, -d):
                if dt > 0:
                    speed += accel * dt
                    t += dt
        prev = coord
    return t
```

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```
def quadratic_roots(two_a, b, c):
    D = b * b - 2 * two_a * c
    if D >= 0:
        if D > 0:
            r = D ** 0.5
            roots = [(-b + r) / two_a, (-b - r) / two_a]
        else:
            roots = [-b / two_a]
    else:
        roots = []
    return roots
```

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```
def main(n=6):
    (y1, y2) = (1.0, 0.0)
    (x1, x2) = (0.0, 1.0)
    coords = [  # initial guess: straight line
        [x1 + (x2 - x1) * i / n,
         y1 + (y2 - y1) * i / n]
        for i in range(n + 1)
    ]
    f = falling_time
    h = 0.00001
    while newton_minimizer_step(f, coords, h) > 0.01:
        print(coords)

if __name__ == '__main__':
    main()
```

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(Demo)

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$$\epsilon_{k+1} = \epsilon_k - \frac{f(x_k)}{f'(x_k)} \qquad \text{(error step)}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{\cancel{f(x_\infty)} + \epsilon_k f'(x_\infty) + \frac{1}{2}\epsilon_k^2 f''(x_\infty) + \cdots}{f'(x_\infty) + \epsilon_k f''(x_\infty) + \cdots} \qquad \text{(Taylor series)}$$

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As $\epsilon_k \to 0$, the "$\cdots$" terms are quickly dominated. Therefore:

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As $\epsilon_k \to 0$, the "$\cdots$" terms are quickly dominated. Therefore:

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Analysis

Error analysis: If x_∞ is the root and $\epsilon_k = x_k - x_\infty$ is the error, then:

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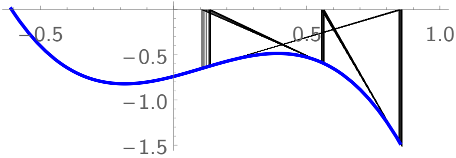

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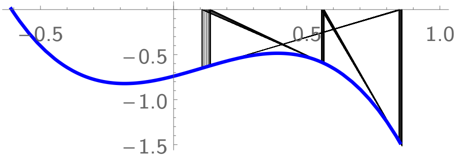

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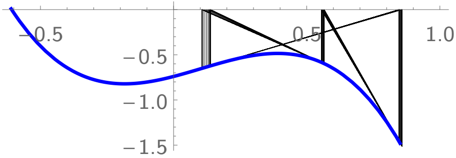

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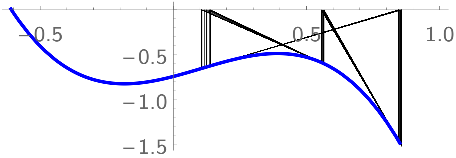

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- Obviously this is possible... just put the root far away
- Therefore Newton-Raphson *can't* be foolproof

Final thoughts

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Can't tell why?

What happens if `y1 = 0.5` instead of `y1 = 1.0`?

Addendum 1

There's *never* a one-size-fits-all solution

[3] Equivalent to *complex-differentiability*: $f'(x) = \lim_{h \to 0} \big(f(x+h) - f(x)\big)/h,\ h \in \mathbb{C}$.

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  - Also said to be *holomorphic*[3]

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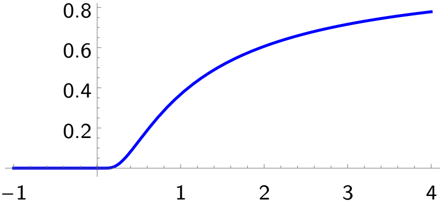

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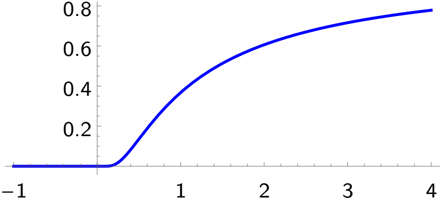

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However, knowing derivatives **would** be enough for **analytic** functions!

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Fun facts:

[4] If $\sum_{k=0}^{n} a_{n-k}x^{k} = 0$ then $|x| \leq 2\max_k \sqrt[k]{|a_k/a_n|}$

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  - Polynomials can approximate real-world functions very well
  - Pretty much **everything** about polynomials has been solved
    - Global root bound (Fujiwara[4]) $\implies$ you know where to **start**
    - Minimal root separation (Mahler) $\implies$ you know when to **stop**
    - Guaranteed root-finding (Sturm) $\implies$ you can **binary-search**

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So, polynomials are *really* nice!

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- Difficult for $\geq 2^{\text{nd}}$ derivatives (need *multicomplex numbers*)

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P.S.: Did you prefer the coding part? Or the math part?