61A Lecture 6

Announcements

Recursive Functions

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Demo

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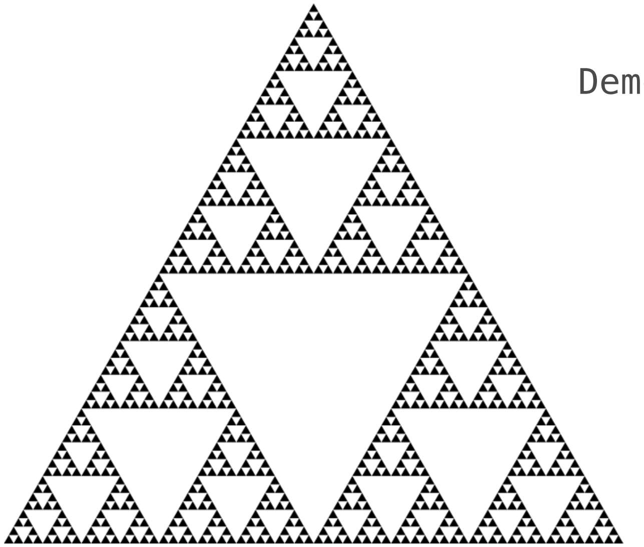

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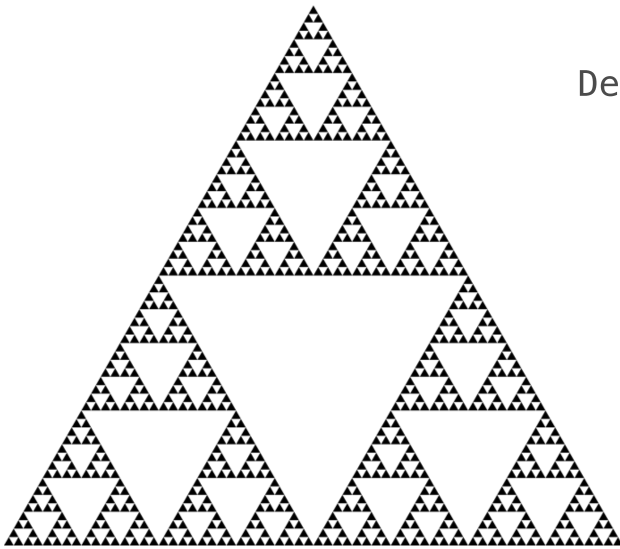

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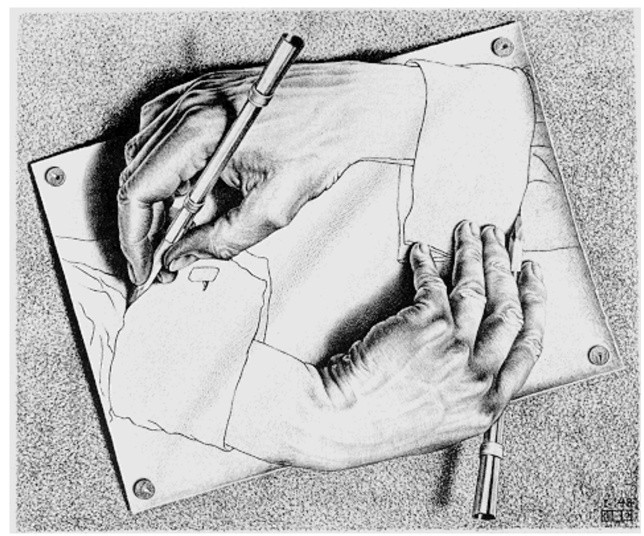

Drawing Hands, by M. C. Escher (lithograph, 1948)

Digit Sums

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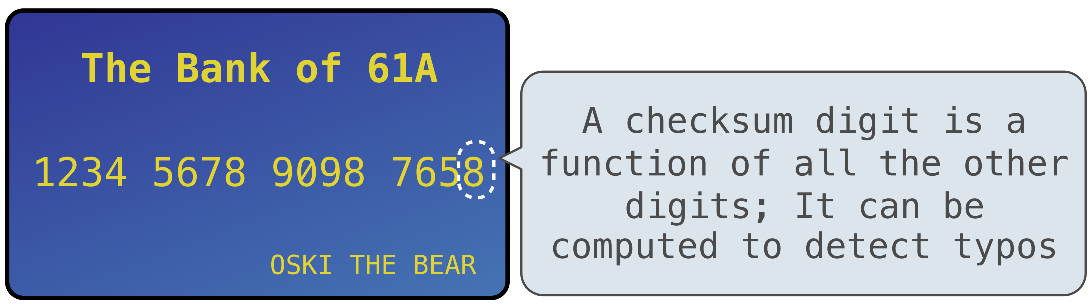

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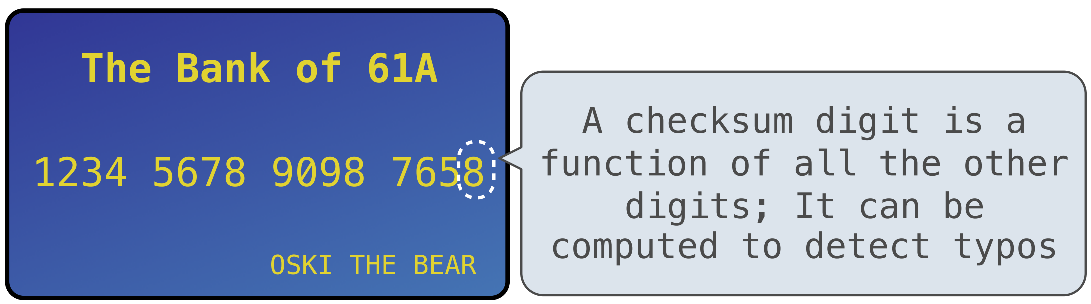

- Credit cards actually use the Luhn algorithm, which we'll implement after `sum_digits`

The Problem Within the Problem

The sum of the digits of 6 is 6.

Likewise for any one-digit (non-negative) number (i.e., < 10).

The sum of the digits of 2016 is

201 6

Sum of these digits + This digit

That is, we can break the problem of summing the digits of 2016 into a smaller instance of the same problem, plus some extra stuff.

We call this recursion

Sum Digits Without a While Statement

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Recursion in Environment Diagrams

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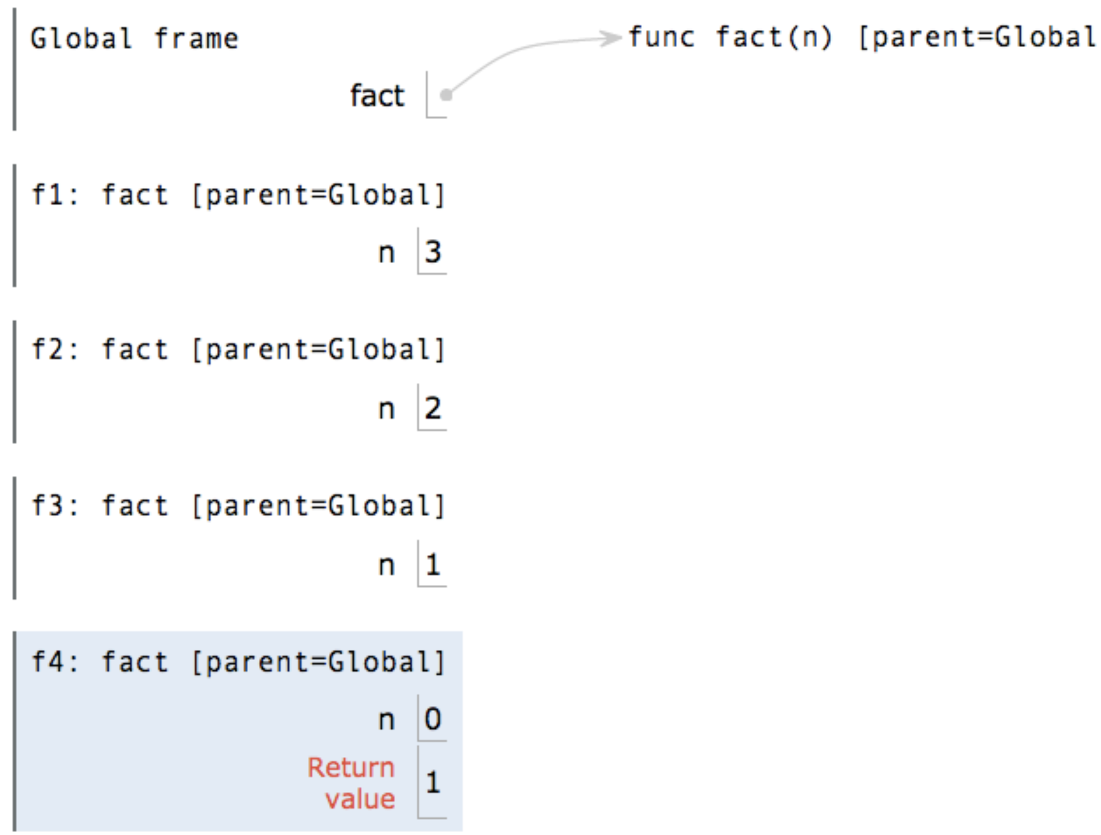

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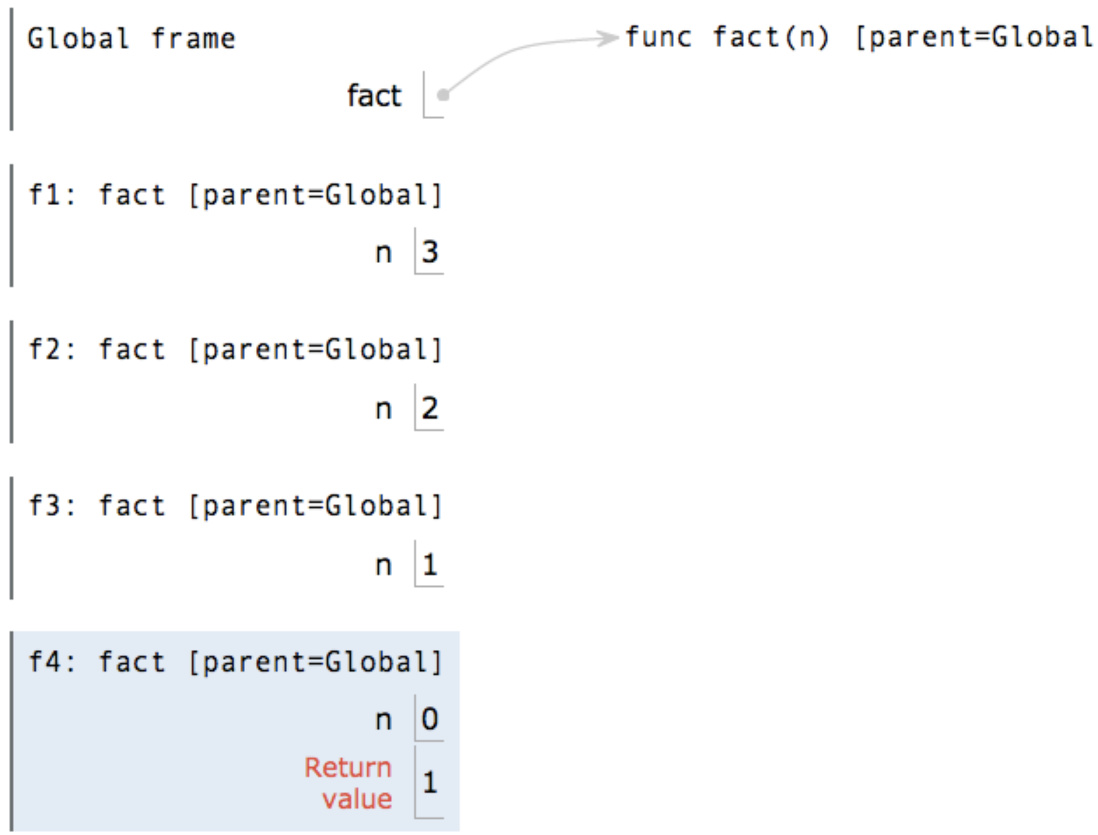

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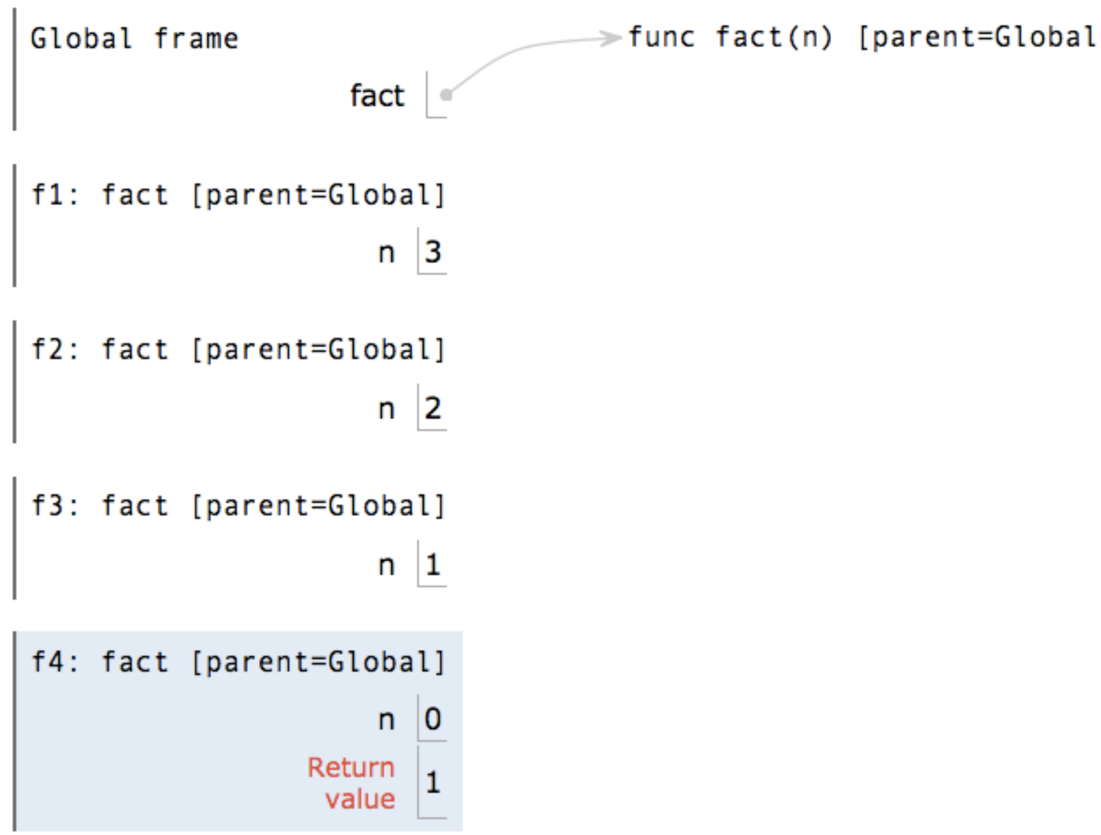

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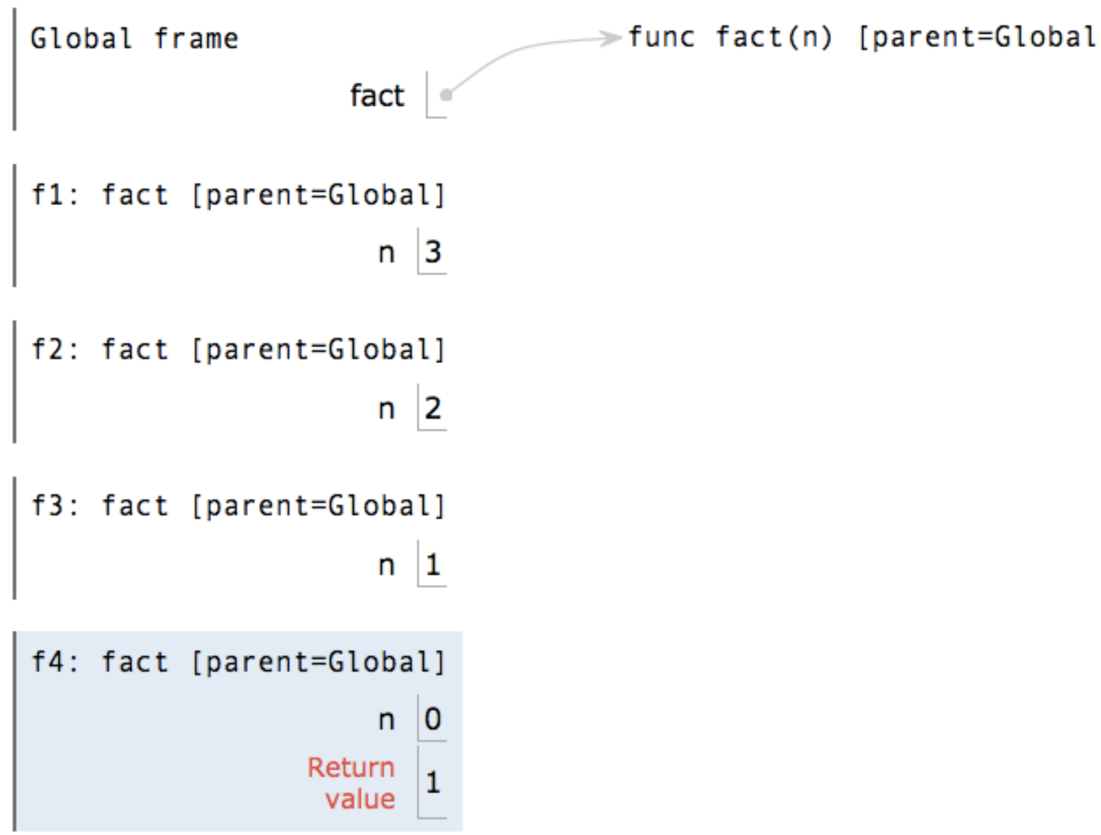

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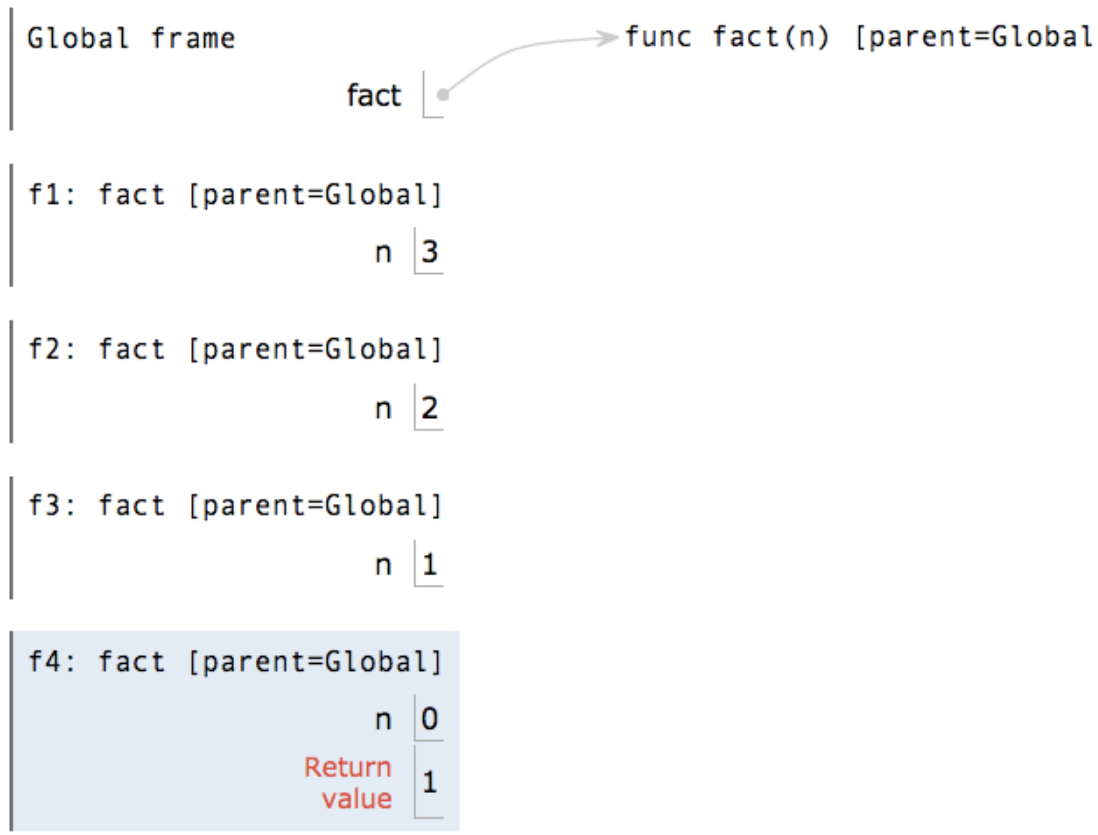

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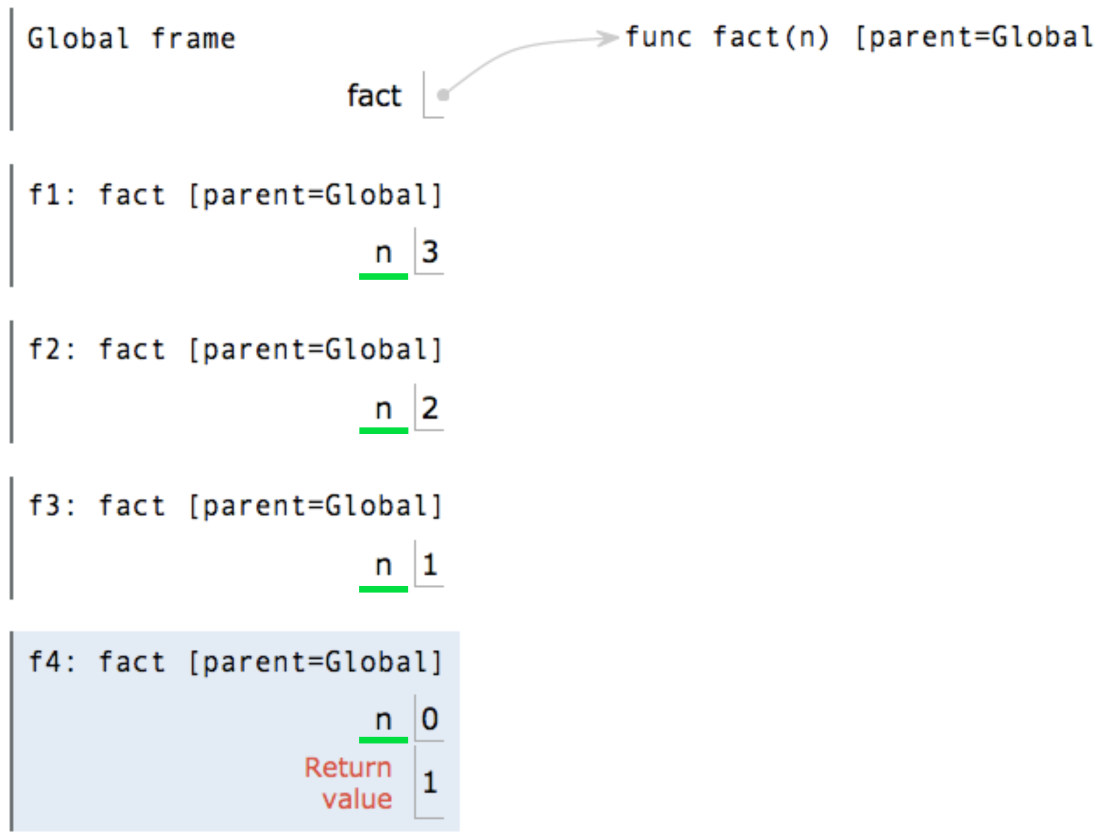

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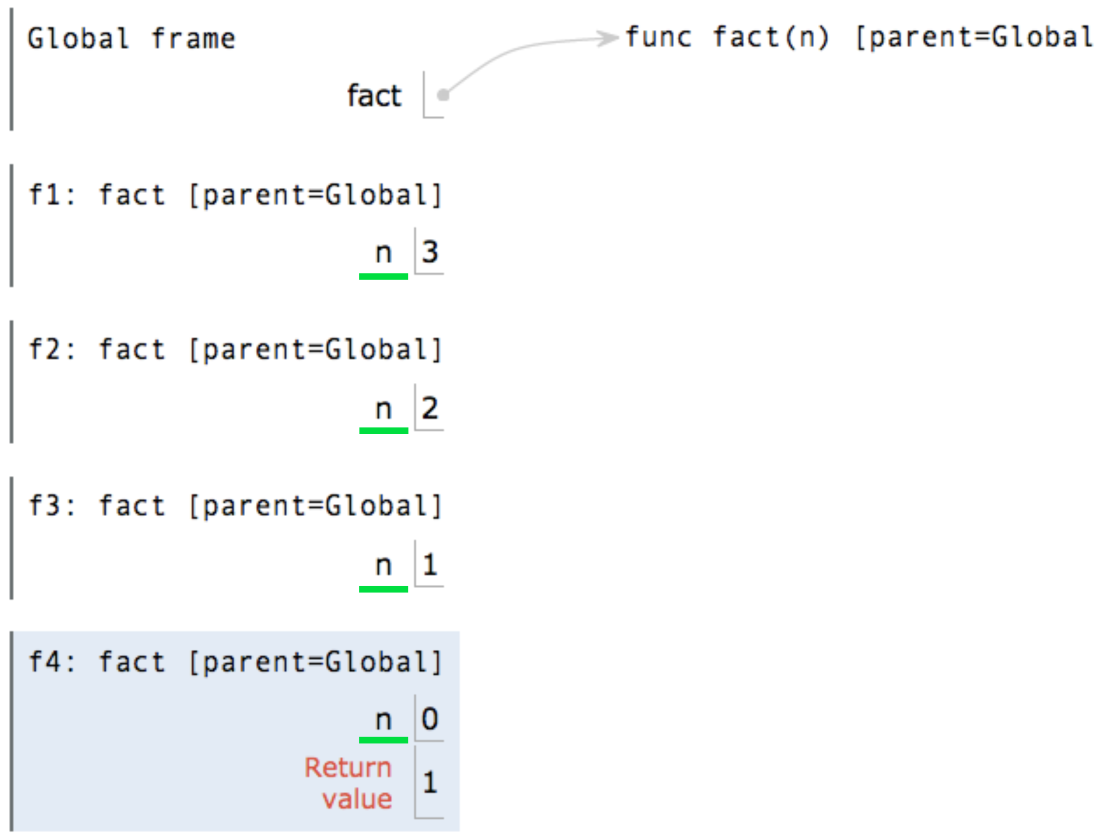

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- Each call to **fact** solves a simpler problem than the last: smaller **n**

Interactive Diagram

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Verifying Recursive Functions

The Recursive Leap of Faith

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Photo by Kevin Lee, Preikestolen, Norway

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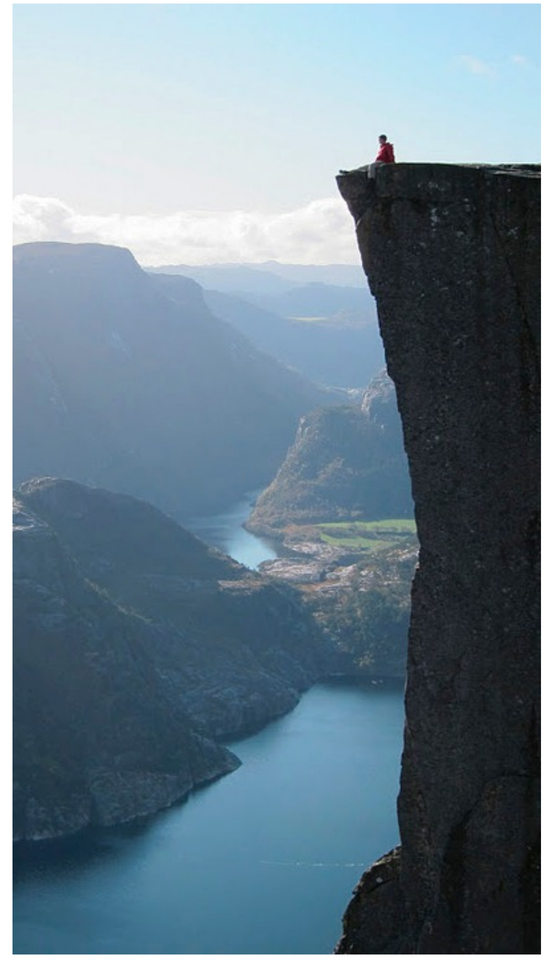

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Mutual Recursion

The Luhn Algorithm

The Luhn Algorithm

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Used to verify credit card numbers
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Recursion and Iteration

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def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = split(n)
        return sum_digits(all_but_last) + last
```

A partial sum

What's left to sum

Converting Recursion to Iteration

Can be tricky: Iteration is a special case of recursion.

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(Demo)

Converting Iteration to Recursion

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def sum_digits_iter(n):
    digit_sum = 0
    while n > 0:
        n, last = split(n)
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def sum_digits_rec(n, digit_sum):
    if n == 0:
        return digit_sum
    else:
        n, last = split(n)
        return sum_digits_rec(n, digit_sum + last)
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Updates via assignment become...

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...arguments to a recursive call