Growth

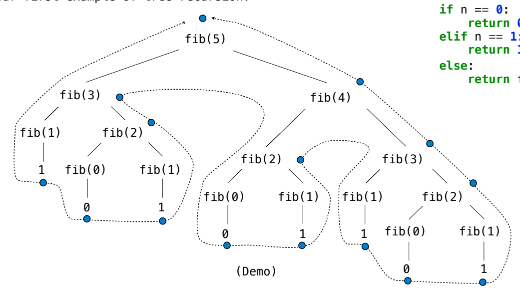
Announcements

Measuring Efficiency

Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:

```python
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```



(Demo)

http://en.wikipedia.org/wiki/File:Fibonacci.jpg

Memoization

Memoization

Idea: Remember the results that have been computed before

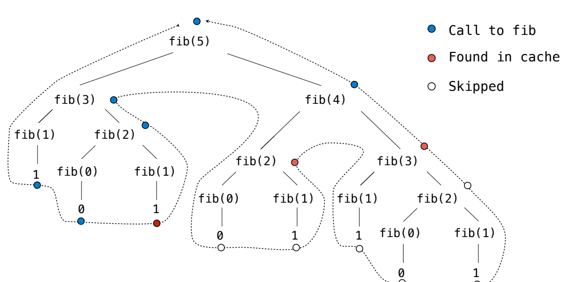
```python
def memo(f):
    cache = {}
    def memoized(n):
        if n not in cache:
            cache[n] = f(n)
        return cache[n]
    return memoized
```

Keys are arguments that map to return values

Same behavior as f, if f is a pure function

(Demo)

Memoized Tree Recursion



Space

The Consumption of Space

Which environment frames do we need to keep during evaluation?

At any moment there is a set of active environments

Values and frames in active environments consume memory

Memory that is used for other values and frames can be recycled

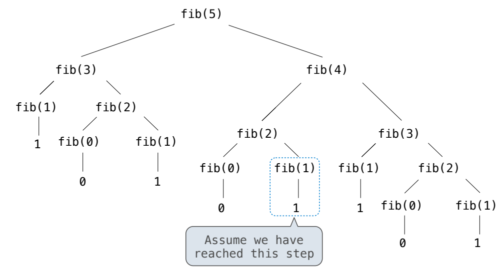
Active environments:

- Environments for any function calls currently being evaluated
- Parent environments of functions named in active environments

(Demo)

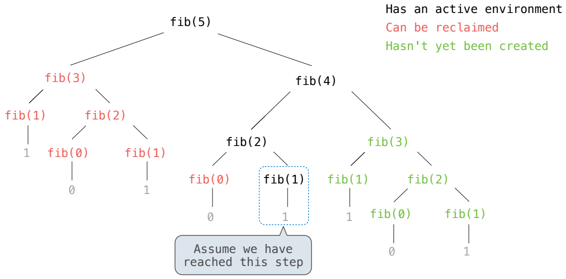
Interactive Diagram

Fibonacci Space Consumption



Fibonacci Space Consumption

Has an active environment
Can be reclaimed
Hasn't yet been created



Time

Comparing Implementations

Implementations of the same functional abstraction can require different resources

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

def factors(n):	**Time (number of divisions)**
Slow: Test each k from 1 through n	n
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	Greatest integer less than $\sqrt{n}$

Question: How many time does each implementation use division? (Demo)

Orders of Growth

Order of Growth

A method for bounding the resources used by a function by the "size" of a problem

n: size of the problem

R(n): measurement of some resource used (time or space)

$$R(n) = \Theta(f(n))$$

means that there are positive constants $\mathbf{k_1}$ and $\mathbf{k_2}$ such that

$$k_1 \cdot f(n) \leq \boxed{R(n)} \leq k_2 \cdot f(n)$$

for all **n** larger than some minimum **m**

Order of Growth of Counting Factors

Implementations of the same functional abstraction can require different amounts of time

Problem: How many factors does a positive integer n have?

A factor k of n is a positive integer that evenly divides n

def factors(n):	**Time**	**Space**
Slow: Test each k from 1 through n	$\Theta(n)$	$\Theta(1)$
Fast: Test each k from 1 to square root n For every k, n/k is also a factor!	$\Theta(\sqrt{n})$	$\Theta(1)$

Assumption: integers occupy a fixed amount of space

(Demo)

Exponentiation

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \cdot b^{n-1} & \text{otherwise} \end{cases}$$

```
def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)

def square(x):
    return x*x

def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)
```

	Time	Space
`exp`	$\Theta(n)$	$\Theta(n)$
`exp_fast`	$\Theta(\log n)$	$\Theta(\log n)$

Comparing Orders of Growth

Properties of Orders of Growth

Constants: Constant terms do not affect the order of growth of a process

$\Theta(n)$ $\quad\Theta(500 \cdot n)$ $\quad\Theta(\frac{1}{500} \cdot n)$

Logarithms: The base of a logarithm does not affect the order of growth of a process

$\Theta(\log_2 n)$ $\quad\Theta(\log_{10} n)$ $\quad\Theta(\ln n)$

Nesting: When an inner process is repeated for each step in an outer process, multiply the steps in the outer and inner processes to find the total number of steps

```
def overlap(a, b):
    count = 0
    for item in a:          Outer: length of a
        if item in b:       Inner: length of b
            count += 1
    return count
```

If a and b are both length **n**, then overlap takes $\Theta(n^2)$ steps

Lower-order terms: The fastest-growing part of the computation dominates the total

$\Theta(n^2)$ $\quad\Theta(n^2 + n)$ $\quad\Theta(n^2 + 500 \cdot n + \log_2 n + 1000)$

Comparing orders of growth (n is the problem size)

$\Theta(b^n)$ Exponential growth. Recursive fib takes $\Theta(\phi^n)$ steps, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61828$

Incrementing the problem scales R(n) by a factor

$\Theta(n^2)$ Quadratic growth. E.g., overlap

Incrementing n increases R(n) by the problem size n

$\Theta(n)$ Linear growth. E.g., slow factors or exp

$\Theta(\sqrt{n})$ Square root growth. E.g., factors_fast

$\Theta(\log n)$ Logarithmic growth. E.g., exp_fast

Doubling the problem only increments R(n).

$\Theta(1)$ Constant. The problem size doesn't matter