CS61B Lecture #15: Complexity

What Are the Questions?

- Cost is a principal concern throughout engineering:

    "An engineer is someone who can do for a dime what any fool can do for a dollar."

- Cost can mean

    – Operational cost (for programs, time to run, space requirements).
    – Development costs: How much engineering time? When delivered?
    – Costs of failure: How robust? How safe?

- Is this program fast enough? Depends on:

    – *For what purpose;*
    – *What input data.*

- How much space (memory, disk space)?

    – Again depends on what input data.

- How will it scale, as input gets big?

Enlightening Example

Problem: Scan a text corpus (say 10^7 bytes or so), and find and print the 20 most frequently used words, together with counts of how often they occur.

- Solution 1 (Knuth): Heavy-Duty data structures

  – Hash Trie implementation, randomized placement, pointers galore, several pages long.

- Solution 2 (Doug McIlroy): UNIX shell script:

  ```
  tr -c -s '[:alpha:]' '[\n*]' < FILE | \
  sort | \
  uniq -c | \
  sort -n -r -k 1,1 | \
  sed 20q
  ```

- Which is better?

  – #1 is much faster,
  – but #2 took 5 minutes to write and processes 20MB in 1 minute.
  – I pick #2.

- In most cases, anything will do: Keep It Simple.

Cost Measures (Time)

- Wall-clock or execution time

  – You can do this at home:

  ```
  time java FindPrimes 1000
  ```

  – Advantages: easy to measure, meaning is obvious.

  – Appropriate where time is critical (real-time systems, e.g.).

  – Disadvantages: applies only to specific data set, compiler, machine, etc.

- Number of times certain statements are executed:

  – Advantages: more general (not sensitive to speed of machine).

  – Disadvantages: doesn't tell you actual time, still applies only to specific data sets.

- Symbolic execution times:

  – That is, *formulas* for execution times as functions of input size.

  – Advantages: applies to all inputs, makes scaling clear.

  – Disadvantage: practical formula must be approximate, may tell very little about actual time.

Asymptotic Cost

- Symbolic execution time lets us see *shape* of the cost function.

- Since we are approximating anyway, pointless to be precise about certain things:

  - Behavior on small inputs:
    * Can always pre-calculate some results.
    * Times for small inputs not usually important.
  - Constant factors (as in "off by factor of 2"):
    * Just changing machines causes constant-factor change.

- How to abstract away from (i.e., ignore) these things?

Handy Tool: Order Notation

- Idea: Don't try to produce *specific* functions that specify size, but rather *families* of *similar* functions.

- Say something like "f is bounded by g if it is in g's family."

- For any function $g(x)$, the functions $2g(x)$, $1000g(x)$, or for any $K > 0$, $K \cdot g(x)$, all have the same "shape". So put all of them into g's family.

- Any function $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some constant M) has g's shape "except for small values." So put all of these in g's family.

- If we want upper limits, throw in all functions that are everywhere $\leq$ some member of g's family. Call this family $O(g)$ or $O(g(n))$.

- Or, if we want lower limits, throw in all functions that are everywhere $\geq$ some member of g's family. Call this family $\Omega(g)$.

- Finally, define $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed* by members of g's family.
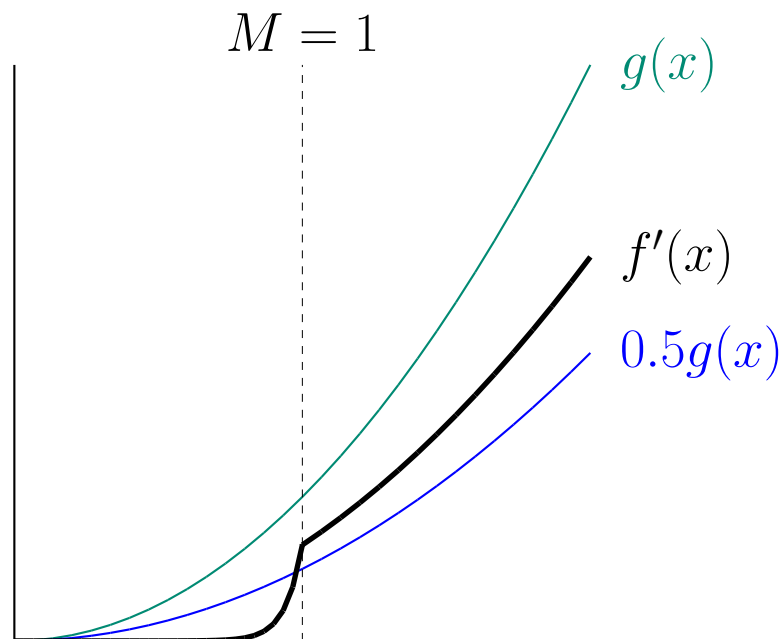
Big Oh

- Goal: Specify bounding from above.
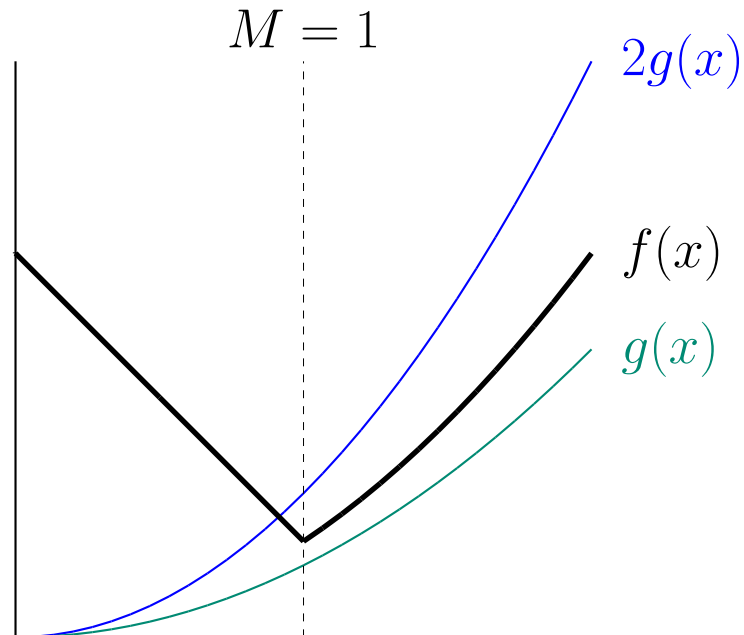
- Here, $f(x) \leq 2g(x)$ as long as $x > 1$,
- So $f(x)$ is in g's upper-bound family, written

$$f(x) \in O(g(x)),$$

- ...even though $f(x) > g(x)$ everywhere.

Big Omega

- Goal: Specify bounding from below:

$$M = 1$$

$$g(x)$$

$$f'(x)$$

$$0.5g(x)$$

- Here, $f'(x) \geq \frac{1}{2}g(x)$ as long as $x > 1$,
- So $f'(x)$ is in g's lower-bound family, written

$$f'(x) \in \Omega(g(x)),$$

- ...even though $f(x) < g(x)$ everywhere.
- In this case, also have $f'(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$, so

$$f(x), f'(x) \in \Theta(g(x)).$$

Why It Matters

- Computer scientists often talk as if constant factors didn't matter at all, only the difference of $\Theta(N)$ vs. $\Theta(N^2)$.

- In reality they do, but at some point, constants always get swamped.

n	$16\lg n$	$\sqrt{n}$	n	$n\lg n$	n^2	n^3	2^n
2	16	1.4	2	2	4	8	4
4	32	2	4	8	16	64	16
8	48	2.8	8	24	64	512	256
16	64	4	16	64	256	4,096	65,636
32	80	5.7	32	160	1024	32,768	4.2×10^9
64	96	8	64	384	4,096	262,144	1.8×10^{19}
128	112	11	128	896	16,384	2.1×10^9	3.4×10^{38}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1,024	160	32	1,024	10,240	1.0×10^6	1.1×10^9	1.8×10^{308}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{20}	320	1024	1.0×10^6	2.1×10^7	1.1×10^{12}	1.2×10^{18}	$6.7\times10^{315,652}$

Some Intuition on Meaning of Growth

- How big a problem can you solve in a given time?

- In the following table, left column shows time in microseconds to solve a given problem as a function of problem size N.

- Entries show the *size of problem* that can be solved in a second, hour, month (31 days), and century, for various relationships between time required and problem size.

- N = problem size

Time (μsec) for problem size N	Max N Possible in			
	1 second	1 hour	1 month	1 century
$\lg N$	10^{300000}	$10^{1000000000}$	$10^{8 \cdot 10^{11}}$	$10^{9 \cdot 10^{14}}$
N	10^6	$3.6 \cdot 10^9$	$2.7 \cdot 10^{12}$	$3.2 \cdot 10^{15}$
$N \lg N$	63000	$1.3 \cdot 10^8$	$7.4 \cdot 10^{10}$	$6.9 \cdot 10^{13}$
N^2	1000	60000	$1.6 \cdot 10^6$	$5.6 \cdot 10^7$
N^3	100	1500	14000	150000
2^N	20	32	41	51

Using the Notation

- Can use this order notation for any kind of real-valued function.

- We will use them to describe cost functions. Example:

```java
/** Find position of X in list L, or -1 if not found. */
int find(List L, Object X) {
    int c;
    for (c = 0; L != null; L = L.next, c += 1)
        if (X.equals(L.head)) return c;
    return -1;
}
```

- Choose representative operation: number of `.equals` tests.

- If N is length of L, then loop does *at most* N tests: *worst-case time* is N tests.

- In fact, total # of instructions executed is roughly proportional to N in the worst case, so can also say worst-case time is $O(N)$, regardless of units used to measure.

- Use $N > M$ provision (in defn. of $O(\cdot)$) to handle empty list.

Be Careful

- It's also true that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ also: Big-Oh bounds are loose.

- The worst-case time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* mean that the loop *always* takes time N, or even $K \cdot N$ for some K.

- Instead, we are just saying something about the *function* that maps N into the *largest possible* time required to process an array of length N.

- To say as much as possible about our worst-case time, we should try to give a Θ bound: in this case, we can: $\Theta(N)$.

- But again, that still tells us nothing about *best-case* time, which happens when we find X at the beginning of the loop. Best-case time is $\Theta(1)$.

Effect of Nested Loops

- Nested loops often lead to polynomial bounds:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = 0; j < A.length; j += 1)
        if (i != j && A[i] == A[j])
            return true;
return false;
```

- Clearly, time is $O(N^2)$, where $N = $ A.length. *Worst-case time* is $\Theta(N^2)$.

- Loop is inefficient though:

```java
for (int i = 0; i < A.length; i += 1)
    for (int j = i+1; j < A.length; j += 1)
        if (A[i] == A[j]) return true;
return false;
```

- Now worst-case time is proportional to

$$N - 1 + N - 2 + \ldots + 1 = N(N-1)/2 \in \Theta(N^2)$$

(so asymptotic time unchanged by the constant factor).

Recursion and Recurrences: Fast Growth

- Silly example of recursion. In the worst case, both recursive calls happen:

```java
/** True iff X is a substring of S */
boolean occurs(String S, String X) {
    if (S.equals(X)) return true;
    if (S.length() <= X.length()) return false;
    return
        occurs(S.substring(1), X) ||
        occurs(S.substring(0, S.length()-1), X);
}
```

- Define $C(N)$ to be the worst-case cost of `occurs(S,X)` for `S` of length N, `X` of fixed size N_0, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \leq N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

- So $C(N)$ grows exponentially:

$$C(N) = 2C(N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0} \cdot 1 + 1) + \ldots + 1$$

$$= 2^{N-N_0} + 2^{N-N_0-1} + 2^{N-N_0-2} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$$

Binary Search: Slow Growth

```
/** True X iff is an element of S[L .. U]. Assumes
 *  S in ascending order, 0 <= L <= U-1 < S.length. */
boolean isIn(String X, String[] S, int L, int U) {
    if (L > U) return false;
    int M = (L+U)/2;
    int direct = X.compareTo(S[M]);
    if (direct < 0) return isIn(X, S, L, M-1);
    else if (direct > 0) return isIn(X, S, M+1, U);
    else return true;
}
```

- Here, worst-case time, $C(D)$, (as measured by # of string comparisons), depends on size $D = U - L + 1$.

- We eliminate `S[M]` from consideration each time and look at half the rest. Assume $D = 2^k - 1$ for simplicity, so:

$$C(D) = \begin{cases} 0, & \text{if } D \leq 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases}$$
$$= \underbrace{1 + 1 + \ldots + 1}_{k} + 0$$
$$= k = \lceil \lg D \rceil \in \Theta(\lg D)$$

Another Typical Pattern: Merge Sort

```
List sort(List L) {
    if (L.length() < 2) return L;
    Split L into L0 and L1 of about equal size;
    L0 = sort(L0);  L1 = sort(L1);
    return  Merge of L0 and L1
}
```

Merge ("combine into a single or-dered list") takes time proportional to size of its result.

- Assuming that size of L is $N = 2^k$, worst-case cost function, $C(N)$, counting just merge time ($\propto$ # items merged):

$$
\begin{aligned}
C(N) &= \begin{cases} 1, & \text{if } N < 2; \\ 2C(N/2) + N, & \text{if } N \geq 2. \end{cases} \\
&= 2(2C(N/4) + N/2) + N \\
&= 4C(N/4) + N + N \\
&= 8C(N/8) + N + N + N \\
&= N \cdot 1 + \underbrace{N + N + \ldots + N}_{k = \lg N} \\
&= N + N \lg N \in \Theta(N \lg N)
\end{aligned}
$$

- In general, $\Theta(N \lg N)$ for arbitrary N (not just 2^k).