CS61B Lectures #27

Today:

- Merge sorts
- Quicksort

Readings: Today: *DS(IJ)*, Chapter 8; Next topic: Chapter 9.

Merge Sorting

Idea: Divide data in 2 equal parts; recursively sort halves; merge results.

- Already seen analysis: $\Theta(N \lg N)$.
- Good for *external sorting:*
  - First break data into small enough chunks to fit in memory and sort.
  - Then repeatedly merge into bigger and bigger sequences.
- Can merge K sequences of *arbitrary size* on secondary storage using $\Theta(K)$ storage:

```
Data[] V = new Data[K];
For all i, set V[i] to the first data item of sequence i;
while there is data left to sort:
    Find k so that V[k] has data and is smallest;
    Add V[k] to output sequence;
    If there is more data in sequence k, read it into V[k],
        otherwise, clear V[k];
```

Illustration of Internal Merge Sort

For internal sorting, can use a *binomial comb* to orchestrate an iterative merge sort.

- Start with $\lg N + 1$ buckets that can contain lists, initially empty.
- Bucket #k is either empty or contains 2^k sorted items at any time.
- For each item in the input list, turn it into a 1-element list, and merge it into bucket 0 (or simply put it in bucket 0 if that is empty).
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- When all inputs are processed, merge all the buckets into the final list.

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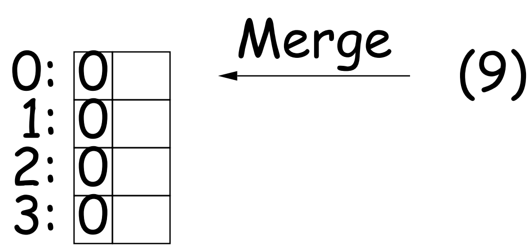

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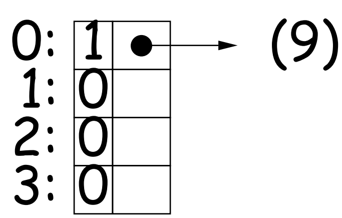

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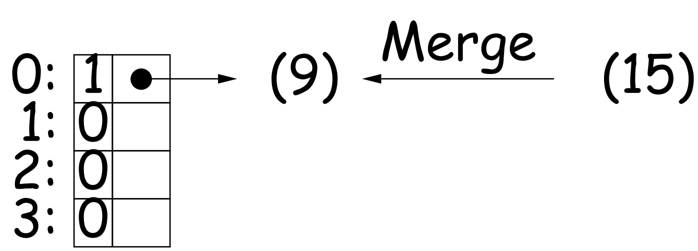

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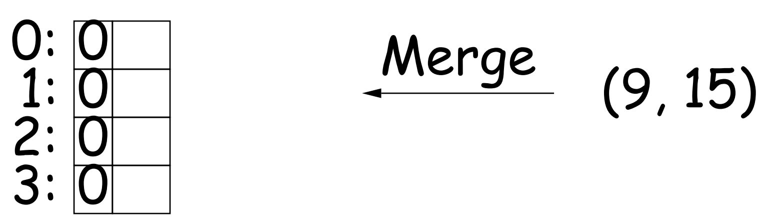

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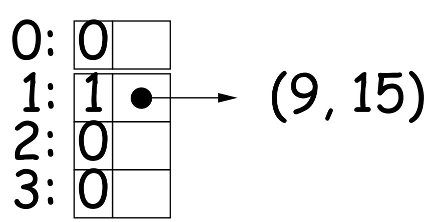

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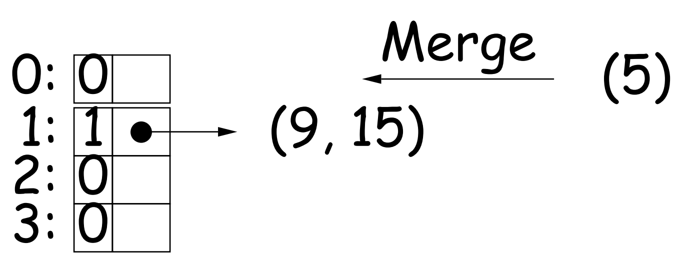

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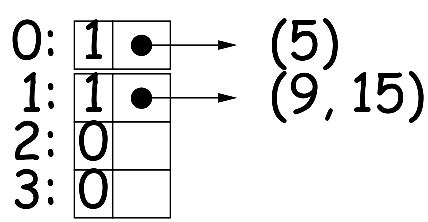

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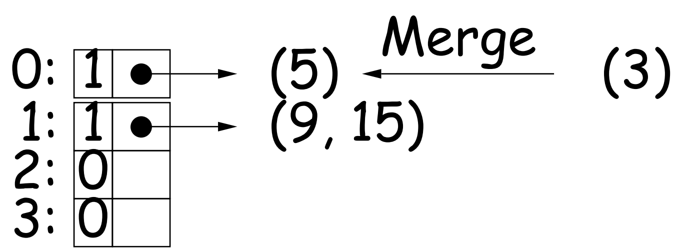

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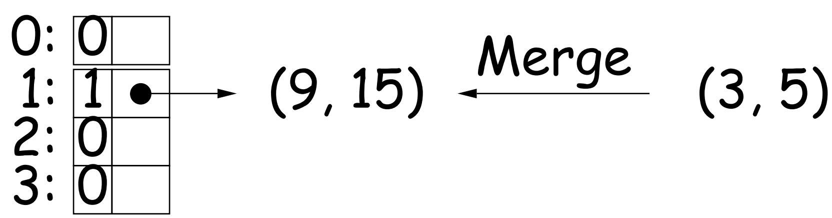

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0:	0	
1:	0	
2:	0	
3:	0	

$\xleftarrow{\text{Merge}}$ (3, 5, 9, 15)

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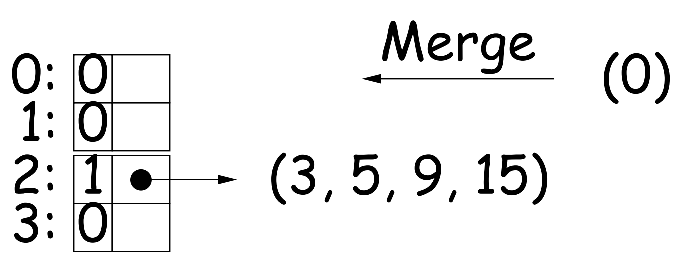

Illustration of Internal Merge Sort (II)

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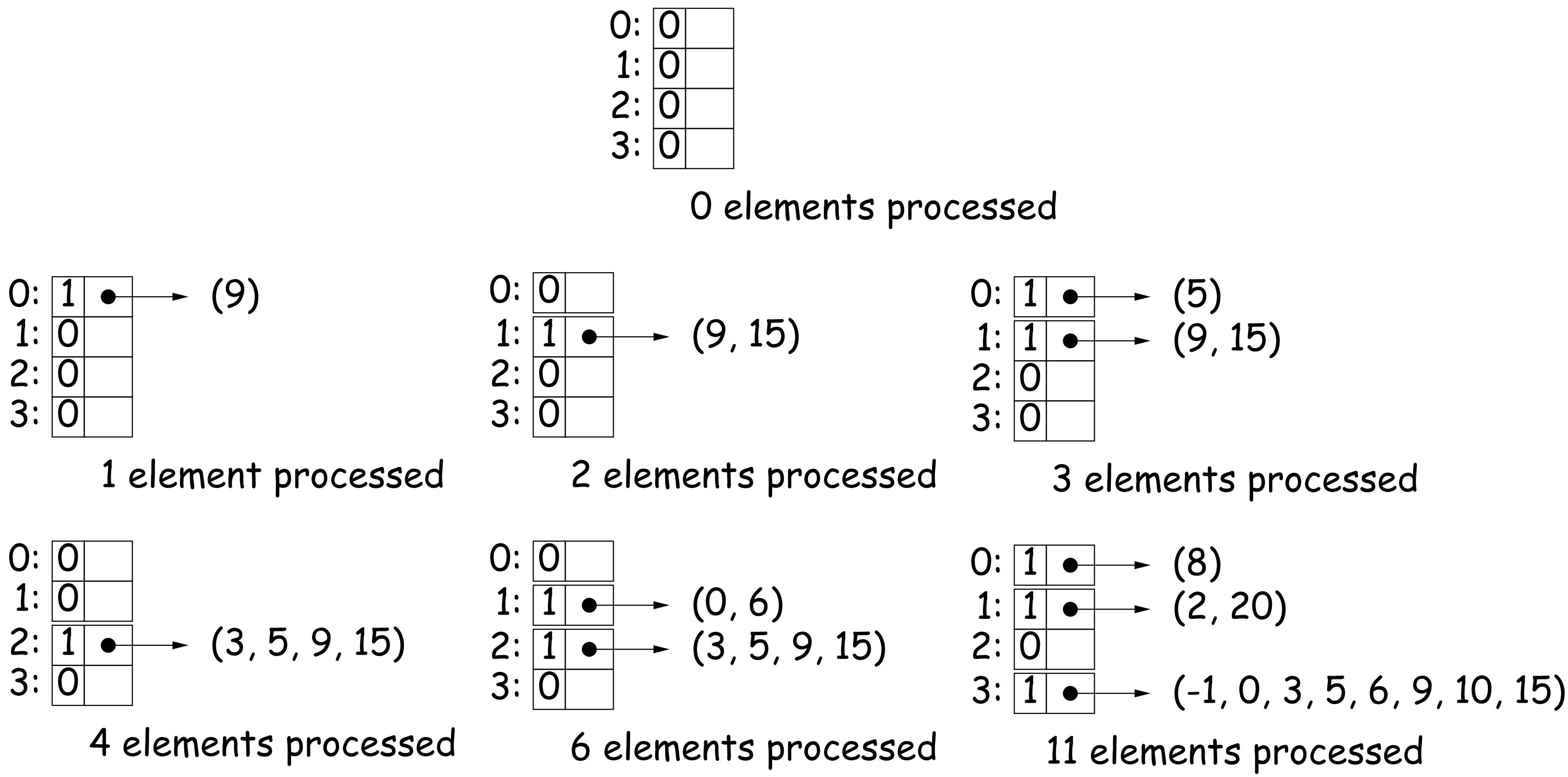

Final Step: Merge all the lists into (-1, 0, 2, 3, 5, 6, 8, 9, 10, 15, 20

Quicksort: Speed through Probability

Idea:

- *Partition* data into pieces: everything $>$ a *pivot* value at the high end of the sequence to be sorted, and everything $\leq$ on the low end.
- Repeat recursively on the high and low pieces.
- For speed, stop when pieces are "small enough" and do insertion sort on the whole thing.
- Reason: insertion sort has low constant factors. By design, no item will move out of its piece [why?], so when pieces are small, #inversions is, too.
- Have to choose pivot well. E.g.: *median* of first, last and middle items of sequence.

Example of Quicksort

- In this example, we continue until pieces are size ≤ 4.
- Pivots for next step are starred. Arrange to move pivot to dividing line each time.
- Last step is insertion sort.

16	10	13	18	-4	-7	12	-5	19	15	0	22	29	34	-1*
-4	-5	-7	-1	18	13	12	10	19	15	0	22	29	34	16*
-4	-5	-7	-1	15	13	12*	10	0	16	19*	22	29	34	18
-4	-5	-7	-1	10	0	12	15	13	16	18	19	29	34	22

- Now everything is "close to" right, so just do insertion sort:

-7	-5	-4	-1	0	10	12	13	15	16	18	19	22	29	34

Performance of Quicksort

- Probabalistic time:
  - If choice of pivots good, divide data in two each time: $\Theta(N \lg N)$ with a good constant factor relative to merge or heap sort.
  - If choice of pivots bad, most items on one side each time: $\Theta(N^2)$.
  - $\Omega(N \lg N)$ in best case, so insertion sort better for nearly ordered input sets.
- Interesting point: randomly shuffling the data before sorting makes $\Omega(N^2)$ time *very* unlikely!

Quick Selection

The Selection Problem: for given k, find kth smallest element in data.

- Obvious method: sort, select element #k, time $\Theta(N \lg N)$.
- If $k \leq$ some constant, can easily do in $\Theta(N)$ time:
  - Go through array, keep smallest k items.
- Get *probably $\Theta(N)$ time* for all k by adapting quicksort:
  - Partition around some pivot, p, as in quicksort, arrange that pivot ends up at dividing line.
  - Suppose that in the result, pivot is at index m, all elements $\leq$ pivot have indicies $\leq m$.
  - If $m = k$, you're done: p is answer.
  - If $m > k$, recursively select kth from left half of sequence.
  - If $m < k$, recursively select $(k - m - 1)$th from right half of sequence.

Selection Example

Problem: Find just item #10 in the sorted version of array:

Initial contents:

51	60	21	-4	37	4	49	10	40*	59	0	13	2	39	11	46	31
0																

Looking for #10 to left of pivot 40:

13	31	21	-4	37	4*	11	10	39	2	0	40	59	51	49	46	60
0																

Looking for #6 to right of pivot 4:

-4	0	2	4	37	13	11	10	39	21	31*	40	59	51	49	46	60
				4												

Looking for #1 to right of pivot 31:

-4	0	2	4	21	13	11	10	31	39	37	40	59	51	49	46	60
									9							

Just two elements; just sort and return #1:

-4	0	2	4	21	13	11	10	31	37	39	40	59	51	49	46	60
									9							

Result: 39

Selection Performance

- For this algorithm, if m roughly in middle each time, cost is

$$
\begin{aligned}
C(N) &= \begin{cases} 1, & \text{if } N = 1, \\ N + C(N/2), & \text{otherwise.} \end{cases} \\
&= N + N/2 + \ldots + 1 \\
&= 2N - 1 \in \Theta(N)
\end{aligned}
$$

- But in worst case, get $\Theta(N^2)$, as for quicksort.
- By another, non-obvious algorithm, can get $\Theta(N)$ worst-case time for all k (take CS170).