CS61B Lecture #35

Today:

- Pseudo-random Numbers (Chapter 11)
- What use are random sequences?
- What *are* "random sequences"?
- Pseudo-random sequences.
- How to get one.
- Relevant Java library classes and methods.
- Random permutations.

Why Random Sequences?

- Choose statistical samples
- Simulations
- Random algorithms
- Cryptography:
  - Choosing random keys and *nonces* (random one-time values used to make messages unique.)
  - Generating streams of random bits (e.g., stream ciphers encrypt messages by xor'ing reproducible streams of pseudo-random bits with the bits of the message.)
- And, of course, games

What Is a "Random Sequence"?

- How about: "a sequence where all numbers occur with equal frequency"?
  - Like 1, 2, 3, 4, ...?
- Well then, how about: "an unpredictable sequence where all numbers occur with equal frequency?"
  - Like 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 3, 4, 4, 0, 1, 1, 1,...?
- Besides, what is wrong with 0, 0, 0, 0, ... anyway? Can't that occur by random selection?

Pseudo-Random Sequences

- Even if definable, a "truly" random sequence is difficult (i.e., slow) for a computer (or human) to produce. Must have some nondeterministic external source. Can use:
  - Periods between radioactive decays.
  - Periods between keystrokes or incoming internet message.
  - Coin flips.
- For most purposes, we need only a sequence that satisfies certain statistical properties, even if deterministic (as is useful for reproducibility).
- Sometimes (e.g., cryptography) we need sequence that is *hard* or *impractical* to predict.
- *Pseudo-random sequence:* deterministic sequence that passes some given set of statistical tests that random sequences (probably) pass.
- For example, look at lengths of *runs:* increasing or decreasing contiguous subsequences.
- Unfortunately, statistical criteria to be used are quite involved. For details, see Knuth, volume 2.

Generating Pseudo-Random Sequences

- Not as easy as you might think.
- Seemingly complex jumbling methods can give rise to bad sequences.
- *Linear congruential method* is a simple method used by Java:

$$\begin{aligned} X_0 &= \textit{arbitrary seed} \\ X_i &= (aX_{i-1} + c) \bmod m, \quad i > 0 \end{aligned}$$

- Usually, m is large power of 2.
- For best results, want $a \equiv 5 \bmod 8$, and a, c, m with no common factors.
- This gives generator with a *period of m* (length of sequence before repetition), and reasonable *potency* (measures certain dependencies among adjacent X_i.)
- Also want bits of a to "have no obvious pattern" and pass certain other tests (see Knuth).
- Java uses $a = 25214903917$, $c = 11$, $m = 2^{48}$, to compute 48-bit pseudo-random numbers. It's good enough for many purposes, but not *cryptographically secure.*

What Can Go Wrong (I)?

Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin.

JOHN VON NEUMANN (1951)

- Short periods, many impossible values: E.g., $a,\ c,\ m$ even.
- Obvious patterns. E.g., just using lower 3 bits of X_i in Java's 48-bit generator, to get integers in range 0 to 7. By properties of modular arithmetic,

$$\begin{aligned} X_i \bmod 8 &= (25214903917 X_{i-1} + 11 \bmod 2^{48}) \bmod 8 \\ &= (5(X_{i-1} \bmod 8) + 3) \bmod 8 \end{aligned}$$

so we have a period of 8 on this generator; sequences like

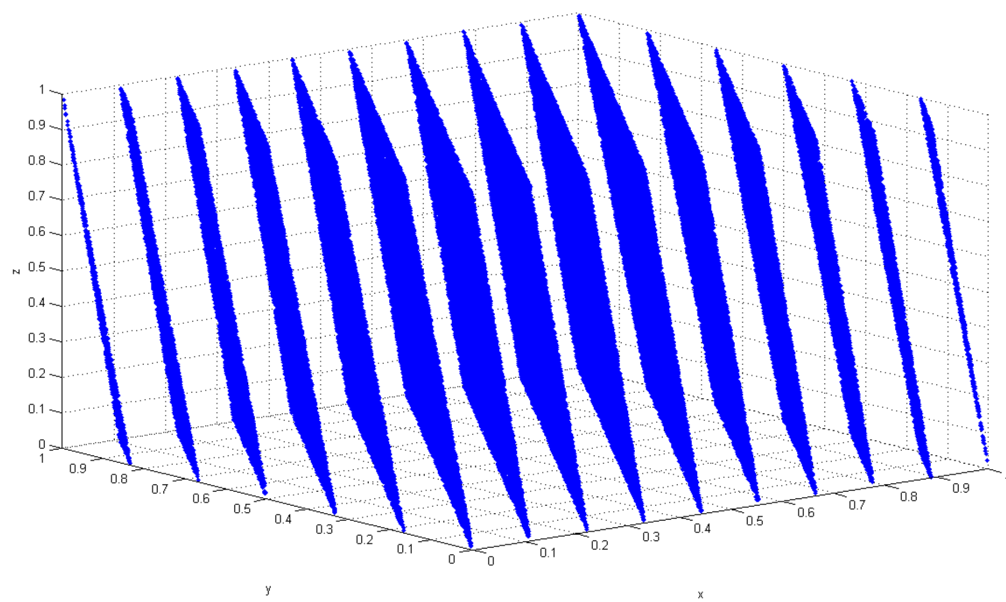
$$0, 1, 3, 7, 1, 2, 7, 1, 4, \ldots$$

are impossible. This is why Java doesn't give you the raw 48 bits.

What Can Go Wrong (II)?

Bad potency leads to bad correlations.

- The infamous IBM generator RANDU: $c = 0$, $a = 65539$, $m = 2^{31}$.
- When RANDU is used to make 3D points: $(X_i/S, X_{i+1}/S, X_{i+2}/S)$, where S scales to a unit cube, ...
- ...points will be arranged in parallel planes with voids between. So "random points" won't ever get near many points in the cube:



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Additive Generators

- Additive generator:

$$X_n = \begin{cases} \textit{arbitary value}, & n < 55 \\ (X_{n-24} + X_{n-55}) \bmod 2^e, & n \geq 55 \end{cases}$$

- Other choices than 24 and 55 possible.
- This one has period of $2^f(2^{55} - 1)$, for some $f < e$.
- Simple implementation with circular buffer:

```
i = (i+1) % 55;
X[i] += X[(i+31) % 55];  // Why +31 (55-24) instead of -24?
return X[i];   /* modulo 2^32 */
```

- where `X[0 .. 54]` is initialized to some "random" initial seed values.

Cryptographic Pseudo-Random Number Generators

- The simple form of linear congruential generators means that one can predict future values after seeing relatively few outputs.
- Not good if you want *unpredictable* output (think on-line games involving money or randomly generated keys for encrypting your web traffic.)
- A *cryptographic pseudo-random number generator (CPRNG)* has the properties that
  - Given k bits of a sequence, no polynomial-time algorithm can guess the next bit with better than 50% accuracy.
  - Given the current state of the generator, it is also infeasible to reconstruct the bits it generated in getting to that state.

Cryptographic Pseudo-Random Number Generator Example

- Start with a good *block cipher*—an encryption algorithm that encrypts blocks of N bits (not just one byte at a time as for Enigma). AES is an example.
- As a seed, provide a key, K, and an initialization value I.
- The j^{th} pseudo-random number is now $E(K, I + j)$, where $E(x, y)$ is the encryption of message y using key x.

Adjusting Range and Distribution

- Given raw sequence of numbers, X_i, from above methods in range (e.g.) 0 to 2^{48}, how to get uniform random integers in range 0 to $n-1$?
- If $n = 2^k$, is easy: use top k bits of next X_i (bottom k bits not as "random")
- For other n, be careful of slight biases at the ends. For example, if we compute $X_i/(2^{48}/n)$ using all integer division, and if $(2^{48}/n)$ gets rounded down, then you can get n as a result (which you don't want).
- If you try to fix that by computing $(2^{48}/(n-1))$ instead, the probability of getting $n-1$ will be wrong.

Adjusting Range (II)

- To fix the bias problems when n does not evenly divide 2^{48}, Java throws out values after the largest multiple of n that is less than 2^{48}:

```
/** Random integer in the range 0 .. n-1, n>0. */
int nextInt(int n) {
  long X = next random long (0 ≤ X < 2^48);
  if (n is 2^k for some k)
      return top k bits of X;

  int MAX = largest multiple of n that is < 2^48;
  while (X_i >= MAX)
      X = next random long (0 ≤ X < 2^48);
  return X_i / (MAX/n);
}
```

Arbitrary Bounds

- How to get arbitrary range of integers (L to U)?
- To get random `float`, x in range $0 \leq x < d$, compute

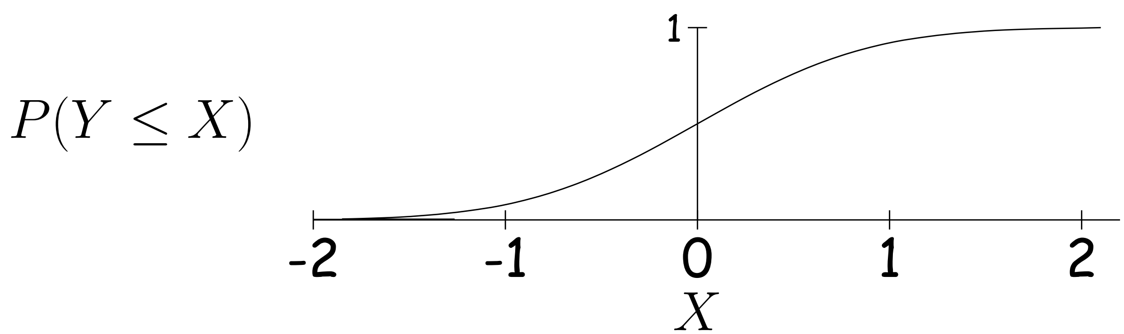
```
return d*nextInt(1<<24) / (1<<24);
```

- Random `double` a bit more complicated: need two integers to get enough bits.

```
long bigRand = ((long) nextInt(1<<26) << 27)
               + (long) nextInt(1<<27);
return d * bigRand / (1L << 53);
```

Generalizing: Other Distributions

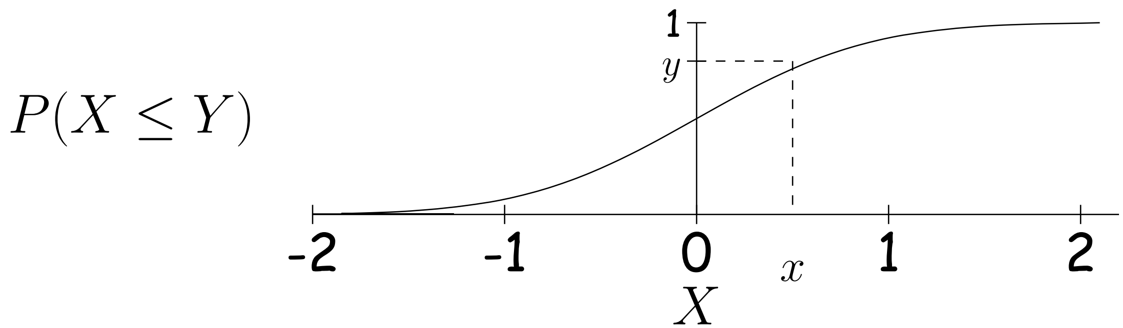
- Suppose we have some desired probability distribution function, and want to get random numbers that are distributed according to that distribution. How can we do this?
- Example: the normal distribution:



- Curve is the desired probability distribution. $P(Y \le X)$ is the probability that random variable Y is $\le X$.

Generalizing: Other Distributions (II)

Solution: Choose y uniformly between 0 and 1, and the corresponding x will be distributed according to P.



Java Classes

- `Math.random()`: random double in $[0..1)$.
- Class `java.util.Random`: a random number generator with constructors:

  Random() generator with "random" seed (based on time).

  Random(seed) generator with given starting value (reproducible).
- Methods

  next(k) k-bit random integer

  nextInt(n) `int` in range $[0..n)$.

  nextLong() random 64-bit integer.

  nextBoolean(), nextFloat(), nextDouble() Next random values of other primitive types.

  nextGaussian() normal distribution with mean 0 and standard deviation 1 ("bell curve").
- `Collections.shuffle`(L, R) for list L and `Random` R permutes L randomly (using R).

Shuffling

- A *shuffle* is a random permutation of some sequence.
- Obvious dumb technique for sorting N-element list:
  - Generate N random numbers
  - Attach each to one of the list elements
  - Sort the list using random numbers as keys.
- Can do quite a bit better:

```
void shuffle(List L, Random R) {
    for (int i = L.size(); i > 0; i -= 1)
        swap elements i-1 and R.nextInt(i) of L;
}
```

- Example:

Swap items	0	1	2	3	4	5
Start	A♣	2♣	3♣	A♡	2♡	3♡
$5 \Longleftrightarrow 1$	A♣	3♡	3♣	A♡	2♡	2♣
$4 \Longleftrightarrow 2$	A♣	3♡	2♡	A♡	3♣	2♣

Swap items	0	1	2	3	4	5
$3 \Longleftrightarrow 3$	A♣	3♡	2♡	A♡	3♣	2♣
$2 \Longleftrightarrow 0$	2♡	3♡	A♣	A♡	3♣	2♣
$1 \Longleftrightarrow 0$	3♡	2♡	A♣	A♡	3♣	2♣

Random Selection

- Same technique would allow us to select N items from list:

```
/** Permute L and return sublist of K>=0 randomly
 *  chosen elements of L, using R as random source. */
List select(List L, int k, Random R) {
  for (int i = L.size(); i+k > L.size(); i -= 1)
    swap element i-1 of L with element
      R.nextInt(i) of L;
  return L.sublist(L.size()-k, L.size());
}
```

- Not terribly efficient for selecting random sequence of K distinct integers from $[0..N)$, with $K \ll N$.

Alternative Selection Algorithm (Floyd)

```
/** Random sequence of K distinct integers
 *  from 0..N-1, 0<=K<=N. */
List<Integer> select(int N, int K, Random R)
{
  ArrayList<Integer> S = new ArrayList<>();

  for (int i = N-K; i < N; i += 1) {
    // All values in S are < i
    int s = R.randInt(i+1); // 0 <= s <= i < N
    if (s == S.get(j) for some j)
      // Insert value i (which can't be there
      // yet) after the s (i.e., at a random
      // place other than the front)
      S.add(j+1, i);
    else
      // Insert random value s (which can't be
      // there yet) at front
      S.add(0, s);
  }
  return S;
}
```

Example

i	s	S
5	4	[4]
6	2	[2, 4]
7	5	[5, 2, 4]
8	5	[5, 8, 2, 4]
9	4	[5, 8, 2, 4, 9]

`selectRandomIntegers(10, 5, R)`