CS70 Fall 2013
Discrete Math and Probability Theory

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Lecture 2: Induction

$$\forall n \in \mathbb{N} \quad \sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Base Case: $n = 0$.

Induction Hypothesis: Suppose $\displaystyle\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Induction Step:

$$\sum_{i=0}^{n+1} i^3 = \sum_{i=0}^{n} i^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= (n+1)^2 \left[\frac{n^2}{4} + \frac{4n+4}{4}\right]$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

Induction on n

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

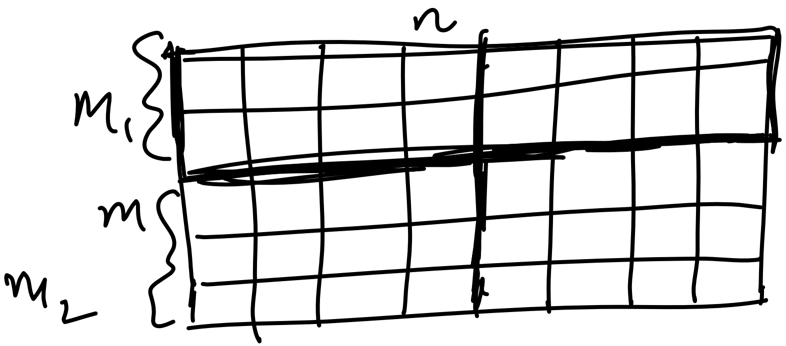
$$\boxed{\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2} = \left(\frac{n(n+1)}{2}\right)^2$$

$$\left(\sum_{i=0}^{n+1} i\right)^2 = \left(\sum_{i=0}^{n} i + n+1\right)^2$$

$$= \underbrace{\left(\sum_{i=0}^{n} i\right)^2}_{\sum_{i=0}^{n} i^3} + 2(n+1)\sum_{i=0}^{n} i + (n+1)^2$$

$$= \sum_{i=0}^{n} i^3 + \underbrace{\cancel{2}(n+1)\left[\frac{n(n+1)}{\cancel{2}}\right] + (n+1)^2}_{(n+1)^2[n+1]}$$

$$= \sum_{i=0}^{n+1} i^3$$



$(m-1) + m(n-1) = mn - 1$

$K = 10$

5×5

4×6

3×7

2×8

1×9

Claim: Make $m \cdot n - 1$ breaks.

Induction on $m+n$.

Base Case: $m+n = 2$ $\quad m = n = 1$

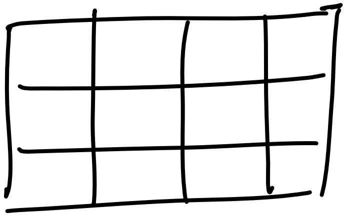
Hypothesis: For $j \leq K$: $\forall m, n \in \mathbb{N} - \{0\}$: $m+n = j$

Need at least $mn - 1$ ...

Step: $m+n = K+1$ $\quad m_1 \times n \quad m_2 \times n \quad \begin{array}{c} m_1 + m_2 \\ \shortparallel \\ m \end{array}$

2 pieces each with $m' + n' \leq K$.

$\cancel{1} + (m_1 n - \cancel{1}) + (m_2 n - 1) = \underline{(m_1 + m_2)} \, \underline{n} - \underline{1}$



Every break gives me extra piece.

Start $\xrightarrow{\#\text{steps.}}$ End

1 piece $\quad$ $\underline{\underline{mn-1}}$ $\quad$ mn pieces.

Ask a grown-up: why don't I like maths?

Who better to answer eight-year-old Connie's question than the mathematics wiz from TV's Countdown?

Rachel Riley
The Guardian, Saturday 2 February 2013

Rachel Riley: 'Hitting a brick wall in maths isn't fun, but look for a new way around a problem and keep trying.' Photograph: Eamonn McCabe/Jaime Turner/GNM Imaging

Countdown's Rachel Riley replies: If you don't enjoy maths, I imagine it's down to one of two reasons: what you're trying is too advanced or too easy. I had a Homer Simpson poster up at university that said, "If something's hard to do, it's not worth doing", which I found funny because my love of maths has always been about the mental challenge. It's supposed to test you and get your brain working.

Hitting a brick wall in maths isn't fun, but look for a new way around a problem and keep trying, and you'll get enormous satisfaction from understanding something that at first has you stumped. Once you've had that eureka moment, you can move on to the next challenge with confidence.

Learning maths is a bit like building a Jenga tower with unlimited pieces. If you try to put pieces on the top too quickly, it'll come crashing down. Mastering the basics is important, as with solid foundations your enjoyment (like a Jenga tower) will grow and grow, with only the sky as your limit.

$\boxed{\forall n \in \mathbb{N}\ P(n)}$ **Strong Induction**

Base Case $P[0]$

Hypothesis: $P(0)$ and $P(1)$ and ..... and $P(n)$

Step: Show $P(n+1)$

Simple (Regular) Induction

Base Case: $P[0]$.

Hypothesis: Assume $P(n)$.

Step: Show $\underline{P(n+1)}$.

$\forall n \in \mathbb{N} \quad n \geq 2 \Rightarrow n = p_1 p_2 \cdots p_k$

p_i prime.

Show

n is a product of primes

$30 = 2 \times 3 \times 5$.

<u>Case 1</u>: n is prime.

<u>Case 2</u>: $n = n_1 \cdot n_2$ $\quad \boxed{n_1, n_2 < n.}$

Induction!!

Sketch

Proof by strong induction on n:

Base case: $n = 2$ ✓

Induction Hypothesis: $2 \leq K \leq n$ K can be written as a product of primes.

Step: Show $n+1$ is a product of primes.

Case 1 $n+1$ prime

Case 2 $n+1 = n_1 \cdot n_2$ $n_1, n_2 \leq n$.

Ind Hypothesis $\Rightarrow$ $n_1 = \prod_{i=1}^{m} p_i$ $n_2 = \prod_{i=1}^{l} q_l$

$$n+1 = n_1 \cdot n_2 = \prod_{i=1}^{m} p_i \prod_{i=1}^{l} q_l .$$

$$1 \times 2 \times 3 \times \cdots \times n = \prod_{i=1}^{n} i$$

$$f(1) \times f(2) \times \cdots \times f(n) = \prod_{i=1}^{n} f(i)$$

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i$$