CS70 Fall 2013

Discrete Math and Probability Theory

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Lecture 3: Induction

Leonardo Fibonacci [1170-1250]

- Decimal notation
- Fibonacci numbers

$F(0) = 0$
$F(1) = 1$
$F(2) = 1$
$F(3) = 2$
$F(4) = 3$
$F(5) = 5$
$F(6) = 8$
$F(7) = 13$
$F(8) = 21$
$F(9) = 34$
$F(10) = 55$
$F(11) = 89$
$F(12) = 144$

Growth of an idealized (biologically unrealistic) rabbit population:
a newly born pair of rabbits, one male, one female, are put in a field;
rabbits are able to mate at the age of one month so that at the end of
its second month a female can produce another pair of rabbits;
rabbits never die and a mating pair always produces one new pair
(one male, one female) every month from the second month on.
The puzzle that Fibonacci posed was: how many pairs will there be
in one year?

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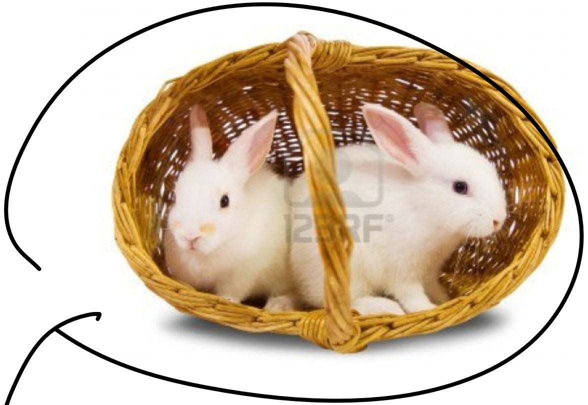
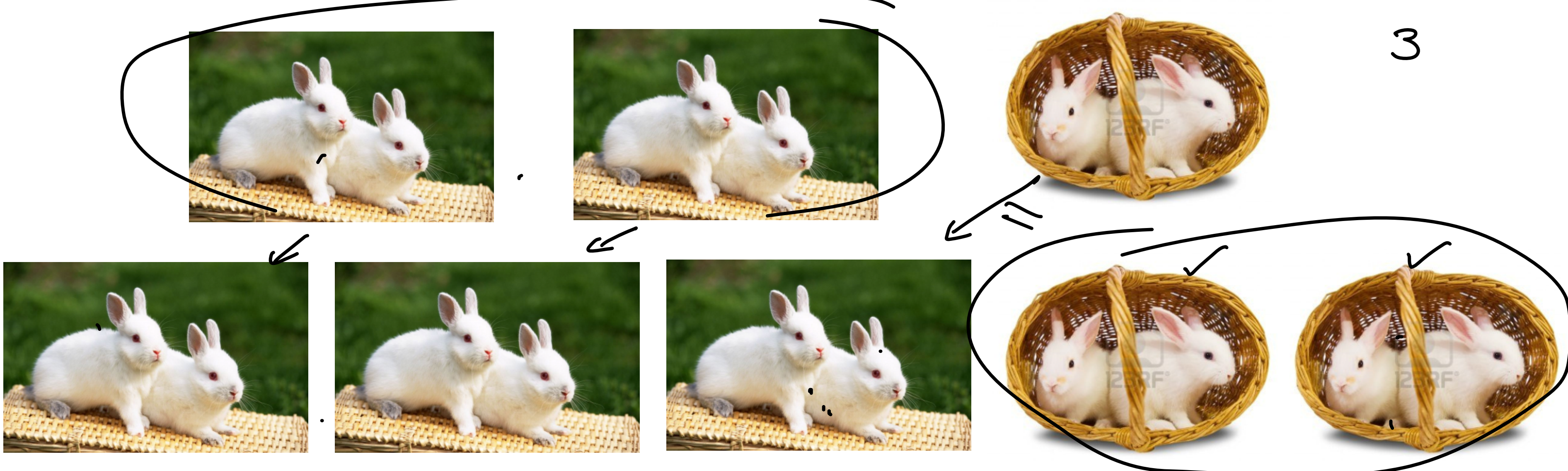


2

3

$$F(n) = \underbrace{F(n-1)}_{\text{\# mature}} + \underbrace{F(n-2)}_{\text{\# babies.}}$$

$$\left.\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \end{aligned}\right\}$$

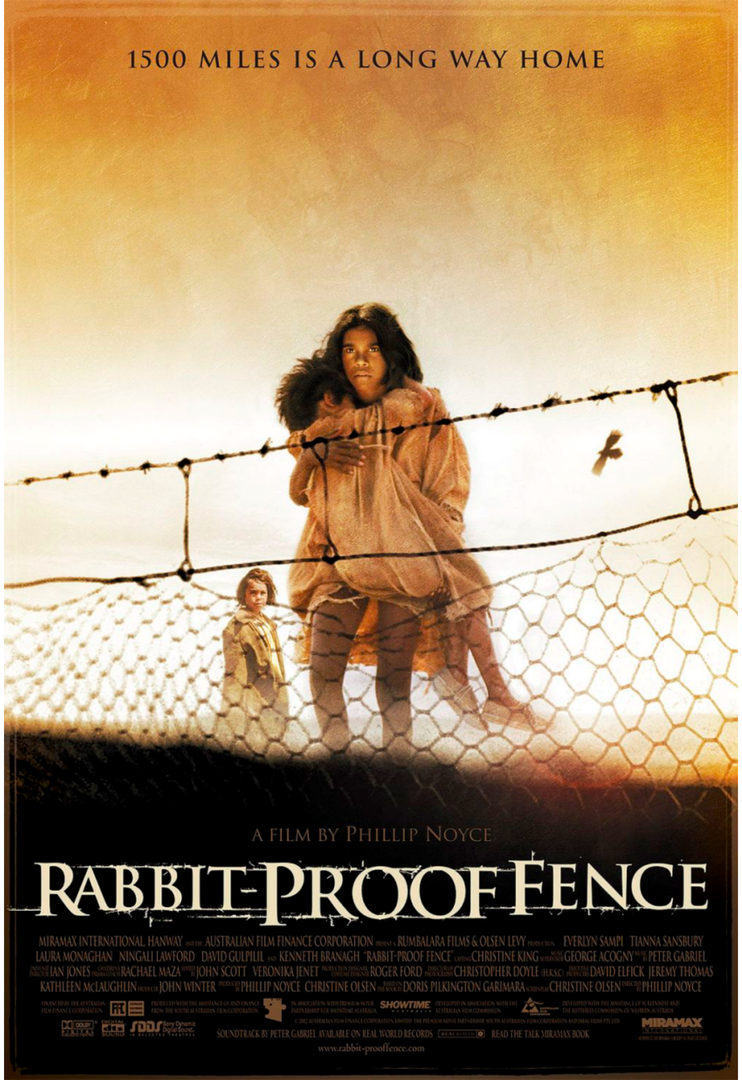


- Release of 24 wild rabbits by Thomas Austin in 1859 Winchelsea, Victoria.

- “The introduction of a few rabbits could do little harm and might provide a touch of home, in addition to a spot of hunting.

- Within 10 years, 2 million could be shot or trapped annually without having any noticeable effect on population.

- In 1907, 2000 mile rabbit-proof fence built to keep rabbits from spreading into Western Australia.



A STRUGGLE for existence inevitably follows from the high rate at which all organic beings tend to increase. Every being, which during its natural lifetime produces several eggs or seeds, must suffer destruction during some period of its life, and during some season or occasional year, otherwise, on the principle of geometrical increase, its numbers would quickly become so inordinately great that no country could support the product. Hence, as more individuals are produced than can possibly survive, there must in every case be a struggle for existence, either one individual with another of the same species, or with the individuals of distinct species, or with the physical conditions of life.

- Charles Darwin in “Origin of Species”

Claim $F(n) \geq 2^{\frac{n-1}{2}}$ for $n \geq 1$ $\quad 2^n$

$F(n) \geq 2^{\frac{n-2}{2}}$

$\approx 10^{10} \approx 2^{32}$

Proof: By induction on n

$10^{80} < 2^{300}$

Base: $n = 1$ $\quad F(n) = 1 \geq 2^{\frac{1-1}{2}} = 2^0 = 1.$

$1 \not> 2^{\frac{2-1}{2}} = 2^{1/2} = \sqrt{2}$

$n=2$ $\quad F(2) = 1$

Hypothesis: $\forall \underset{1}{\underset{\vee}{k}} \leq n$ $\quad F(k) \geq 2^{\frac{n-1}{2}}$

$$F(n+1) = F(n) + F(n-1) \geq 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}}$$

$$= 2^{n/2} \cdot 2^{-1/2} + 2^{n/2} \cdot 2^{-1}$$

$$= 2^{n/2} \left[\frac{1}{\sqrt{2}} + \frac{1}{2} \right] \quad 2^{n/2}$$

$$\geq 2^{n/2} \left[\frac{1}{2} + \frac{1}{2} \right] = 2^{n/2}$$

$$F(n) = F(n-1) + F(n-1)$$

$F(0) = 1$

$F(1) = 2$

$F(2) = 4$

$F(3) = 8$

$\vdots$

$F(n) = 2^n$

$$F(n) = F(n-1) + F(n-2)$$

$$F(n) \gtrsim 2^{n/2}$$

$$= \left(2^{1/2}\right)^n = (1.4\ldots)^n$$

$$\approx \phi^n$$ golden ratio $(1.61)^n$

$n+1$

```
function F(n)
 if n=0 then return 0
 if n=1 then return 1
 else return F(n-1) + F(n-2)
```

$F(n) = F(n-1) + F(n-2)$

$F(0) = 0$

$F(1) = 1$

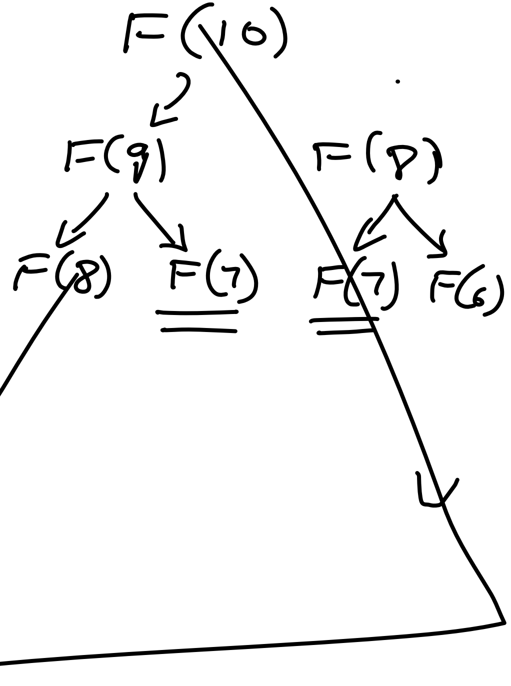
$T(n) \geq F(n)$

Base $T(0) \geq 0$ $\quad T(1) \geq 1 = F(1)$

Hypothesis: $\forall K \leq n \quad T(K) \geq F(K)$.

Step $T(n+1) \geq F(n) + F(n-1)$

$= F(n+1)$



```
function F2(n)
  if n=0 then return 0
  if n=1 then return 1
  a = 1
  b = 0
  for k = 2 to n do
    temp = a
    a = a + b
    b = temp
  return a
```

$\leftarrow F(k-1)$

$\leftarrow F(k-2)$

Fibonacci?

<u>Claim</u> $F_2(n) = F(n)$

<u>Claim</u>: End of loop with k

$a = F(k)$

$b = F(k-1)$.

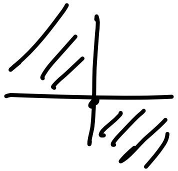
<u>Base</u> $k=2$ $a = F(2)$ $b = F(1)$.

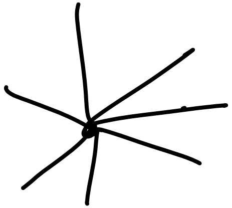
<u>Step</u>: $a \leftarrow a+b = F(k)+F(k-1)$

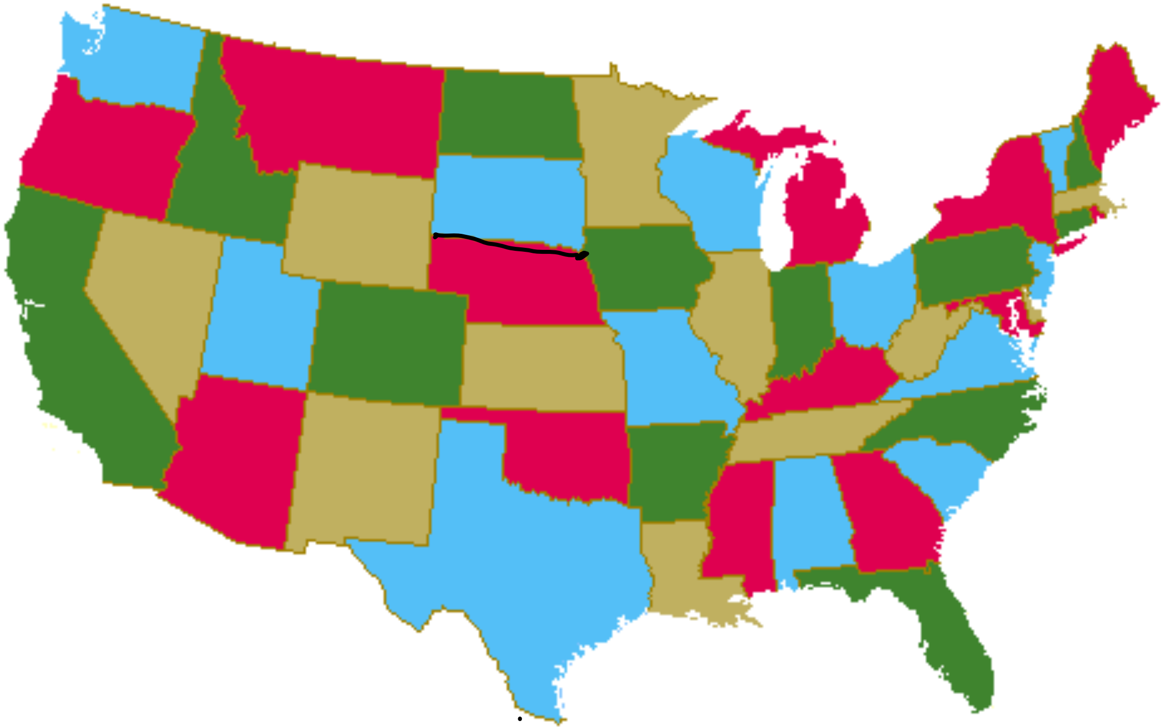
$= F(k+1)$

$b \leftarrow a = F(k)$

Four Color Theorem:

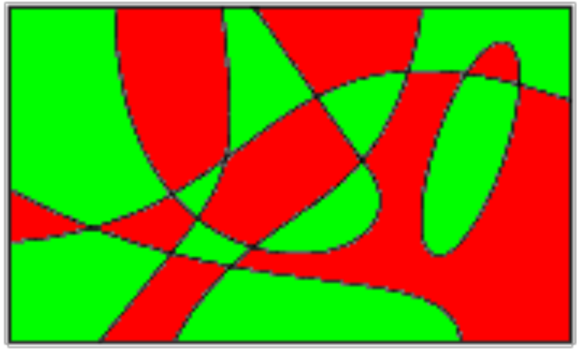


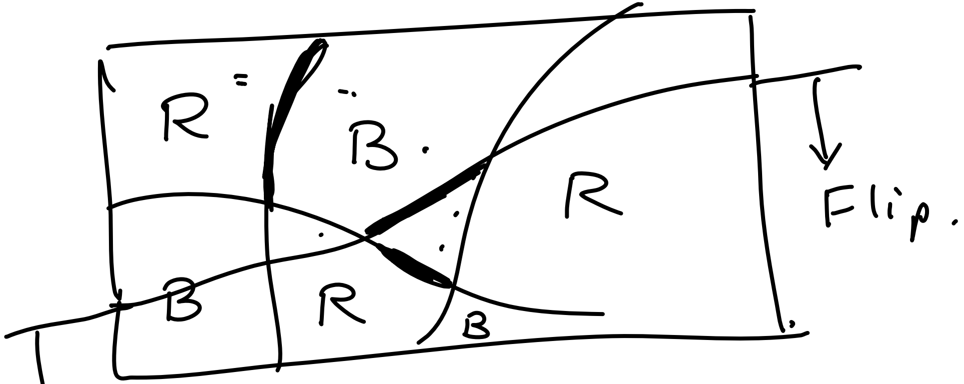




Proved in 1976 by Appel and Haken.

Two Color Theorem:

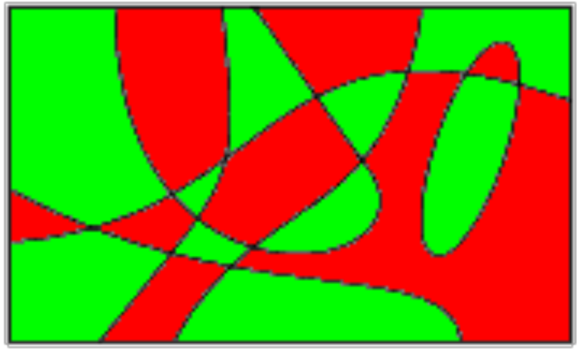




Idea of proof: Add back $k+1^{st}$ line & flip colors on one side

Sketch: boundary is either part of $k+1^{st}$ line or part of original boundary which lies on one side of $k+1^{st}$ line.

Two Color Theorem:



Induction on # lines.

Base: $n = 0$

Hyp: n.

Step: $n+1$

Remove one line. 2 color by Hypothesis

Insert $(n+1)^{st}$ line. Flip colors on one side.

$\underbrace{1+3+5+\cdots+(2k+1)}$ is a perfect square.

Induction on k.

$$1 = 1^2$$
$$1+3 = 4 = 2^2$$
$$1+3+5 = 9$$
$$1+3+5+7 = 16$$

Assume $1+\cdots+2k+1 = n^2$

Prove $\underbrace{1+3+\cdots+2k+1}+2k+3 = m^2$

$$\boxed{n^2+2k+3}$$

$$\overbrace{1+3+\cdots+(2k-1)}^{k} = k^2$$

<u>Step</u>: $$\underbrace{1+3+\cdots+(2k-1)}+(2k+1)$$

$$k^2+2k+1$$

$$=(k+1)^2$$