CS70 Fall 2013
Discrete Math and Probability Theory

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Lecture 4: Induction + Stable Marriage

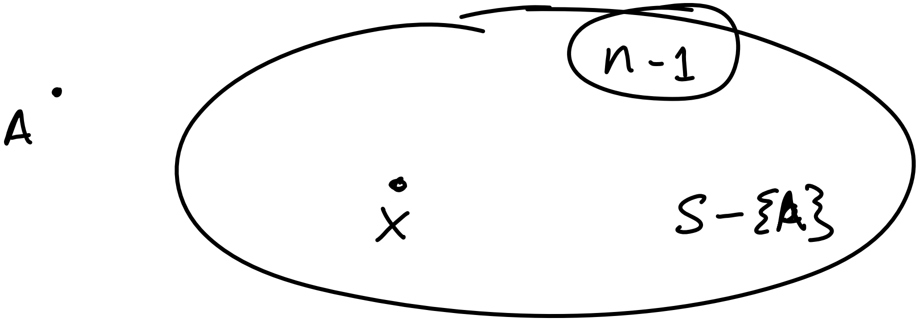
3. A *celebrity* at a party is someone whom everyone knows, yet who knows no one. Suppose that you are at a party with n people. For any pair of people A and B at the party, you can ask A if they know B and receive an honest answer. Show that it is possible to determine whether there is a celebrity at the party, and if so who, by asking at most $3n - 1$ questions.

Your answer should specify your strategy for asking questions, a proof that it always identifies a celebrity if one exists, and a proof that the number of questions is at most $3n - 1$.

(Hint: use one question to identify someone who *isn't* a celebrity, then use recursion.)

$A \longrightarrow B$?

j



S

$S - \{A\}$

CELEB(S)

$$\underbrace{A \to B}_{1}$$

yes $\quad \underbrace{X = \text{CELEB}(S - \{A\})}_{3(n-1)-1}$ Hypothesis.

$$\left.\begin{matrix} A \to X \\ X \to A \end{matrix}\right\} 2$$

Step $\quad 3(n-1) - 1 + 1 + 2 = 3n - 1$

4. **Claim**: If I draw n straight lines on a piece of paper I cannot divide the piece of paper into more than $\frac{n(n+1)}{2}+1$ regions.

Synthesize the core idea of the following proof, and write a 2-3 line "sketch of proof."

Proof: We proceed by induction.

Base Case: If there are no lines then the plane is divided into $1=\frac{0(0+1)}{2}+1$ regions, as desired.

Induction Hypothesis: Suppose that any set of n lines divide the plane into at most $\frac{n(n+1)}{2}+1$ regions.

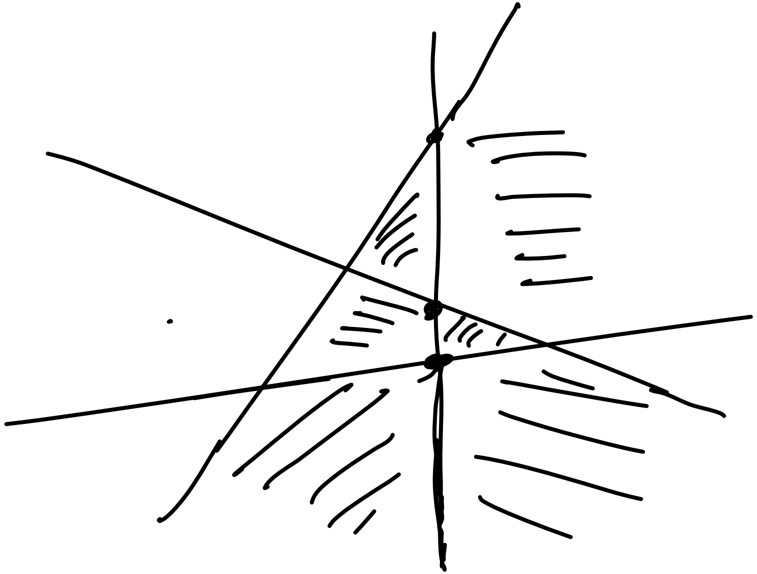
Induction Step: Let S be some set of $n+1$ lines. Let ℓ be an arbitrary line in S, and let T be the rest of S. Let A and B be the part of the sheet of paper on the left and right halves of ℓ. By the inductive hypothesis, T divides the plane into at most $\frac{n(n+1)}{2}+1$ regions, say $R_1, R_2, \ldots$.

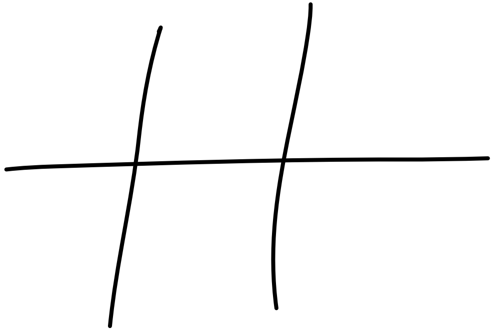
Observe that ℓ can divide each R_i into at most two sub-regions, $R_i \cap A$ and $R_i \cap B$. Moreover, unless ℓ intersects R_i, one of these regions will be empty. Thus the number of new regions created by drawing ℓ is at most the number of regions that ℓ intersects.

Between any two regions that ℓ intersects, there is at least one line which ℓ intersects. Moreover, ℓ intersects each line in T at most once (since any two lines intersect at most once), and there are n lines in T. Thus the number of new regions is at most $\underline{\underline{n+1}}$.

+1

Thus S divides the piece of paper into at most $\frac{n(n+1)}{2}+1+(n+1)=\frac{(n+1)(n+2)}{2}$ regions, as desired.





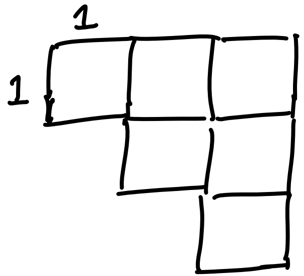
n lines

$n+1^{st}$ line — it intersects each n lines in at most 1 point.

regions = 1+ # intersections.

$$\frac{n(n+1)}{2} + 1 + (n+1) = \frac{(n+1)(n+2)}{2} + 1$$

Build-up Error



$\forall n \geq 1$.

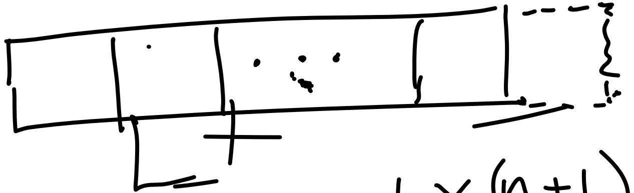
Claim: Only way to create a rectangle using n squares is $1 \times n$ rectangle.

Proof: Induction on n.

Base Case: $n = 1$

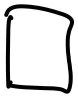
Hypothesis: Only way to create rectangle using n squares is $1 \times n$ rectangle

Step:



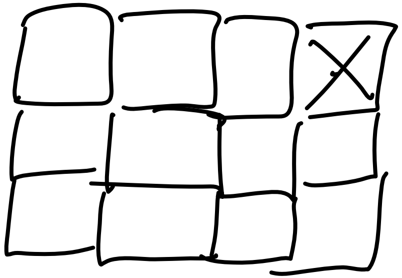
Start with n

Add



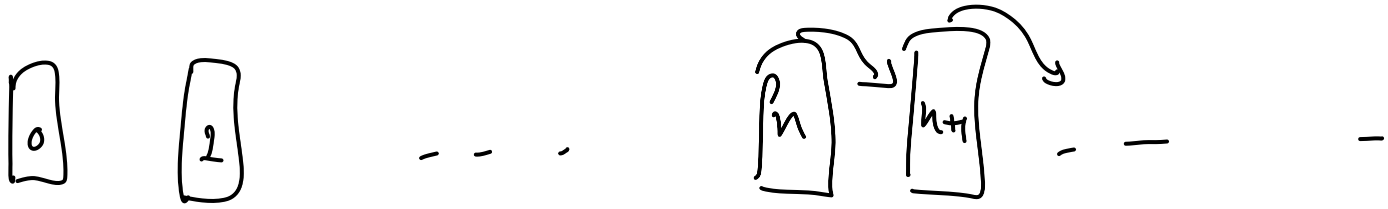
$1 \times (n+1)$

Step: consider $n+1$ squares arrayed in rectangle

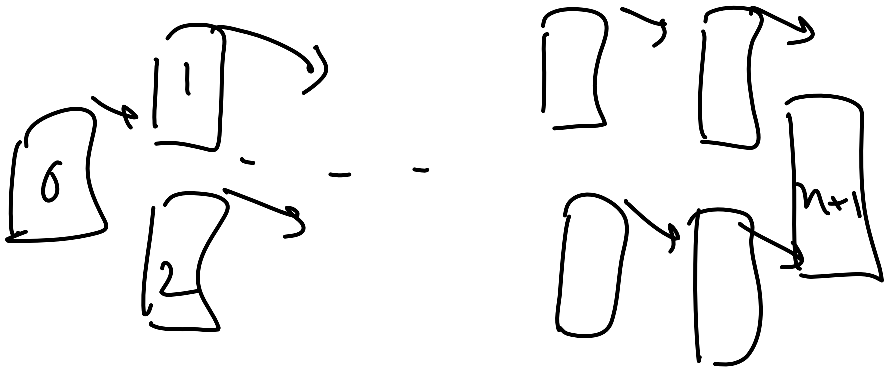


Remove one & apply induction hypothesis

Simple Induction



Strong Induction



Strengthening Induction Hypothesis

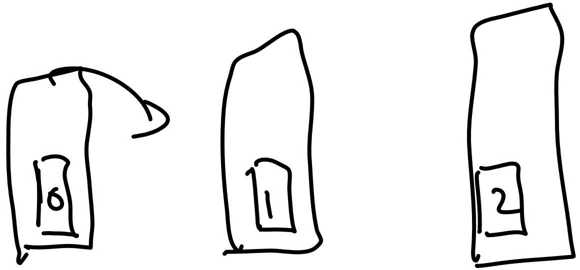
2. Expand the following "idea of a proof" into a complete proof that $\sum_{k=1}^{n} \frac{1}{k^2} \le 2$.

Idea of proof: If $\boxed{\sum_{k=1}^{n} \frac{1}{k^2} \le 2 - \frac{1}{n}}$ then $\sum_{k=1}^{n+1} \frac{1}{k^2} \le 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \le 2 - \frac{1}{n} + \frac{1}{n(n+1)} = 2 - \frac{1}{n+1}$

$$\sum_{k=1}^{n} \frac{1}{k^2} \le 2$$

Step

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \underbrace{\sum_{k=1}^{n} \frac{1}{k^2}}_{\le 2} + \frac{1}{n+1}$$



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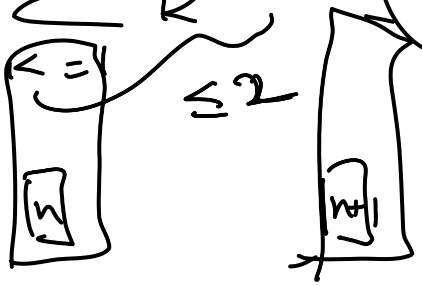

Well ordering principle.

$\boxed{\forall n \in \mathbb{N} \quad P(n)}$

Induction: $P(0)$ and $\forall n \; \underbrace{P(n) \Rightarrow P(n+1)}_{\text{step}}$.

Principle of Induction <u>Axiom</u>

Assume for contradiction $[\forall n \; P(n)]$ is false.
Let K be smallest natural number for which
$P(K)$ is false

$K \neq 0$ $\quad P(K-1)$ is true $\quad P(K-1) \Rightarrow P(K)$

contradiction!!

$a, b \in \mathbb{N}$ $\quad a, b > 0 \quad$ then

$$\exists\, q, r \in \mathbb{N} : \quad a = \underbrace{b\,q}_{\text{quotient}} + \underbrace{r}_{\text{remainder}}$$

$r < b$

Assume not.
Fix b. Find smallest
a : cannot write as
$a = bq + r$.

$a = 13 \quad b = 5$.

$13 = 5 \times 2 + 3$

quotient $\quad$ remainder

$a < b \qquad a = b \cdot 0 + a$

$$\underbrace{a - b}_{a'} = b\,q' + r' \implies a = b\underbrace{(q'+1)}_{q} + \underbrace{r'}_{r}$$

Stable Marriage

- n men and n women, each with a preference list
- Match them to get a “good” pairing.
- Simple and efficient algorithm to achieve this.

Preference Lists

1: A, B, C	**A:** 1, 2, 3
2: B, A, C	**B:** 2, 1, 3
3: B, C, A	**C:** 1, 3, 2

Stable Marriage Algorithm (50s style dating)

Each day:

- Each woman stands on her balcony.
  Each man stands under the balcony of his favorite woman that he hasn't yet crossed off his list and serenades her.
- Women with at least one suitor say to their favorite among them "maybe", and to everyone else "never."
- Every man who hears "never," crosses that woman off his list.

Eventually: If each woman has at most one suitor, stop.

1: A, B, C, D
2: B, A, C, D
3: B, C, A, D
4: B, A, D, C

A: 4, 2, 3, 1
B: 2, 1, 3, 4
C: 1, 2, 3, 4
D: 4, 1, 2, 3

	A	B	C	D
Day 1	1	2, 3, 4		
Day 2	1 4	2	3	
Day 3	4	1 2	3	
Day 4	4	2	1, 3 →	
↓	4	2	1	3

Stable Marriage Algorithm

- Hospital residency match: match interns to hospitals
- Akamai uses modified SMA to match requests to servers
- At least one major dating service uses TMA.

2 From U.S. Win Nobel in Economics

By CATHERINE RAMPELL

Published: October 15, 2012 | 90 Comments

Two Americans, Alvin E. Roth and Lloyd S. Shapley, were awarded the Nobel Memorial Prize in Economic Science on Monday for their work on market design and matching theory, which relate to how people and companies find and select one another in everything from marriage to school choice to jobs to organ donations.

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Their work primarily applies to markets that do not have prices, or at least have strict constraints on prices. The laureates' breakthroughs involve figuring out how to properly assign people and things to stable matches when prices are not available to help buyers and sellers pair up.