CS70 Fall 2013
Discrete Math and Probability Theory

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Lecture 5: Stable Marriage

Stable Marriage Problem

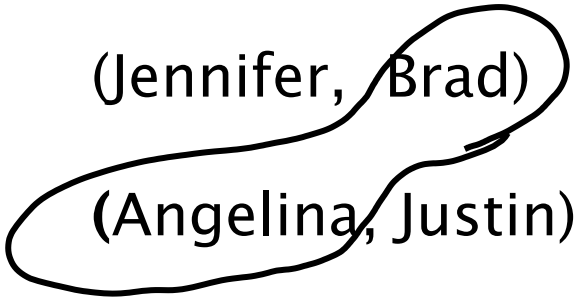
Brad: Angelina, Jennifer
Justin: Angelina, Jennifer

Angelina: Brad, Justin
Jennifer: Brad, Justin

Rogue Couple

Brad: Angelina, Jennifer
Justin: Angelina, Jennifer

Angelina: Brad, Justin
Jennifer: Brad, Justin



Stable Pairing

Brad: Angelina, Jennifer
Justin: Angelina, Jennifer

Angelina: Brad, Justin
Jennifer: Brad, Justin

(Angelina, Brad)

(Jennifer, Justin)

Stable Marriage Algorithm

1: A, B, C, D
2: B, A, C, D
3: B, C, A, D
4: B, A, D, C

A: 4, 2, 3, 1
B: 2, 1, 3, 4
C: 1, 2, 3, 4
D: 4, 1, 2, 3

	A	B	C	D
Day 1	1	2, 3, 4		
Day 2	1, 4	2	3	
Day 3	4	2, 1	3	
Day 4	4	2	3, 1	
Day 5	4	2	3	1

Claim: The stable marriage algorithm terminates in at most $n^2 + 1$ days.

Men ↓ worse every day

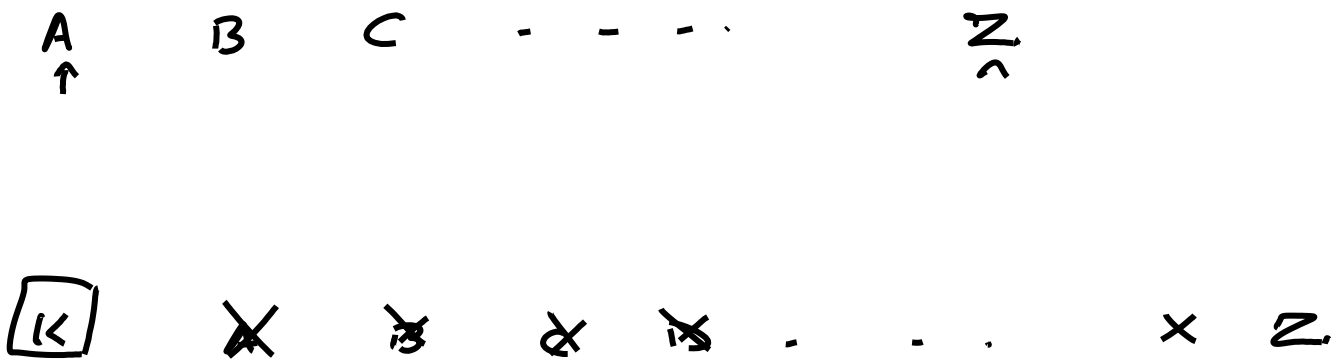
Women ↑ better every day.

Improvement Lemma: If a woman has a man on a string on day j, then on every subsequent day she has someone at least as good on a string.

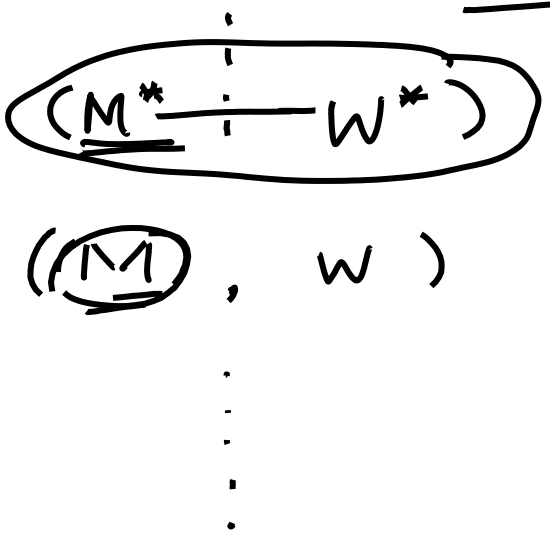
By induction on j.

$j+1$

Claim: The stable marriage algorithm terminates with a pairing of all n men and women.



Theorem: The pairing output by the algorithm is stable.



Assume M prefers W*

Show W* prefers M*

Assume M prefers W*.

M proposes to W* before W

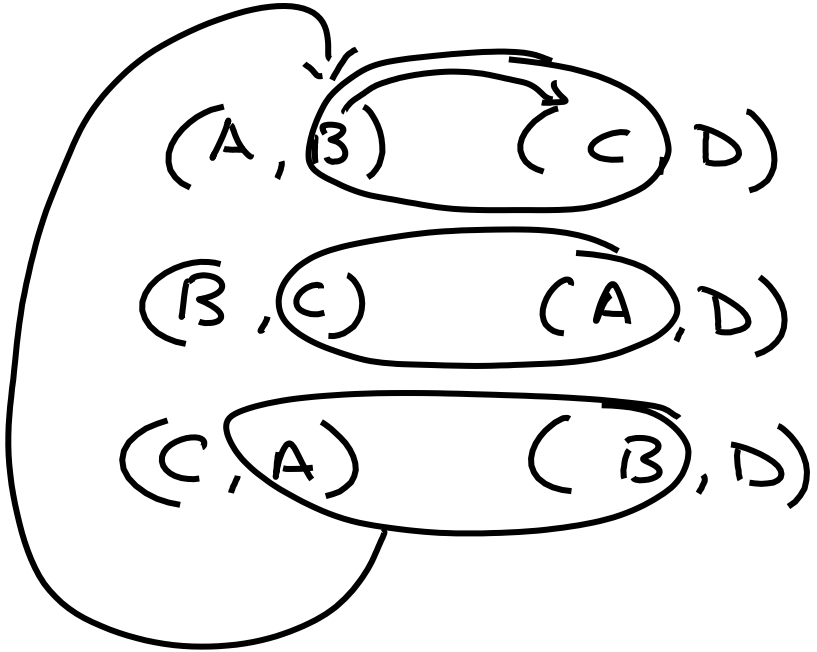
W* rejected M on some day in favor of M'.

By improvement lemma, W* likes M* at least as much as M'.

∴ W* prefers M* to M.

Stable Roommates

Alex: Brad, Charlie, David
Brad: Charlie, Alex, David
Charlie: Alex, Brad, David
David:



Optimality

1: B, A, C, D
2: A, B, D, C
3: D, C, A, B
4: A, B, C, D

A: 1, 2, 3, 4
B: 2, 1, 4, 3
C: 3, 1, 4, 2
D: 1, 4, 3, 2

Stable pairings:
(A, 1) (B, 2) (C, 3) (D, 4)
(A, 2) (B, 1) (C, 3) (D, 4)
(A, 1) (B, 2) (C, 4) (D, 3)
(A, 2) (B, 1) (C, 4) (D, 3)

woman optimal.

man optimal.

"

woman pessimal.

Theorem: The pairing produced by stable marriage algorithm is male-optimal

1	A	·	- - ·
2	B	-	- -
3	C	-	- ·
4	D	-	- -

Run algorithm.

First day: same man rejected by his optimal

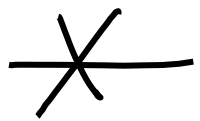
M rejected by w^*

M^*.

$\exists$ stable pairing:

(M, w^*)

(M^*, w')



Theorem: The pairing produced by stable marriage algorithm is female-pessimal.

Moral: Make the first move!

Hospital Residency Match

- Each medical school graduate submits a ranked list of hospitals
- Each hospital submits a ranked list of graduates
- A computer runs a suitable variant of TMA.

Hospital Residency Match

- Intense competition over short supply of residents
- 1940s – appointments made two years before graduation.
- Medical schools stopped releasing info about students until close to graduation.
- Short fuse offers!
- 1950 centralized system.

2 From U.S. Win Nobel in Economics

By CATHERINE RAMPELL
Published: October 15, 2012 | 90 Comments

Two Americans, Alvin E. Roth and Lloyd S. Shapley, were awarded the Nobel Memorial Prize in Economic Science on Monday for their work on market design and matching theory, which relate to how people and companies find and select one another in everything from marriage to school choice to jobs to organ donations.

Enlarge This Image

Their work primarily applies to markets that do not have prices, or at least have strict constraints on prices. The laureates' breakthroughs involve figuring out how to properly assign people and things to stable matches when prices are not available to help buyers and sellers pair up.