Induction

1. Prove that for all $n \in \mathbb{N}$, $5 \mid 6^n - 1$.

2. Consider the Euclidean algorithm:

Algorithm 1: The Euclidean Algorithm

input : A pair of integers $a, b \in \mathbb{N}$.
output: An integer $x \in \mathbb{N}$.
while $a \neq b$ **do**
  if $a > b$ **then**
    Replace a with $a - b$.
  if $b > a$ **then**
    Replace b with $b - a$.
Return a.

   (a) Prove that if k divides a and b, then k divides the output of the Euclidean algorithm. (Hint: use induction on the number of steps taken by the algorithm, starting from the first step.)

   (b) Prove that the output of the Euclidean algorithm divides both a and b. (Hint: use induction on the number of steps taken by the algorithm, but this time start from the last step.)

3. Suppose that you are interested in retrieving an object located in the middle of the desert, n kilometers away. Your car can carry enough gas to travel 3 kilometers, and you have an unlimited supply of spare tanks which you can use to leave deposits of gas throughout the desert. Your starting point has a gas station which can give you as much gas as you need.

   (a) Show that it is possible to retrieve the item and return it to your starting location by driving at most 3^n kilometers. (Hint: strengthen the inductive hypothesis by showing that you can leave some gas behind.)

   (b) What is the minimal number of miles it is possible to travel in order to retrieve the item?

4. The following is a famous theorem due to Ramsey:

   Theorem: For any function f which assigns each pair of integers $a, b \in \mathbb{N}$ one of two colors, there is an infinite set of integers S and a single color such that every pair $a, b \in S$ is assigned that color.

   Using induction, generalize the theorem:

   Theorem: Fix some $k \in \mathbb{N}$. For any function f which assigns each pair of integers $a, b \in \mathbb{N}$ one of k different colors, there is an infinite set of integers S and a single color such that every pair $a, b \in S$ is assigned that color.