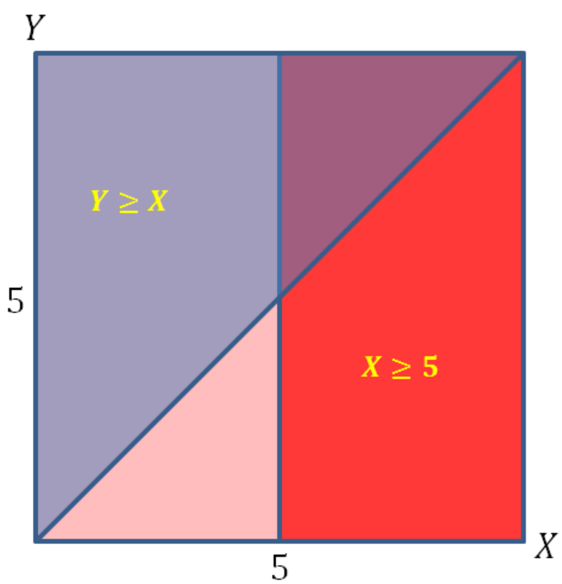
Continuous Probability

Uniform Distribution

You have two spinners, each having a circumference of 10, with values in the range $[0,10)$. If you spin both (independently) and let X be the position of the first spinner and Y be the position of the second spinner, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution: First we write down what we want and expand out the conditioning. $\Pr[X \geq 5 \,|\, Y \geq X] = \frac{\Pr[Y \geq X \cap X \geq 5]}{\Pr[Y \geq X]}$. $\Pr[Y \geq X] = \frac{1}{2}$ by symmetry (why should Y be any more likely to be bigger than X is?). To find $\Pr[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law ($\Pr[A] = \frac{\text{area of } A}{\text{area of } \Omega}$). From the picture, one sees that $\Pr[Y \geq X \cap X \geq 5] = \frac{\frac{5 \cdot 5}{2}}{10 \cdot 10} = \frac{1}{8}$. So $\Pr[X \geq 5 \,|\, Y \geq X] = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$.



Exponential Distribution

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

1. Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

   Solution: Let $X \sim Exp(\frac{1}{50})$ be the time until the bulb is broken. $\Pr[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot e^{-\frac{x}{50}}\right) dx = 1 - e^{-\frac{30}{50}} = 1 - e^{-\frac{3}{5}} \approx .451$

2. Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probablity that the new bulb will last at least 30 days?

   Solution: The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is $\Pr[Y > 30] = 1 - \Pr[Y < 30] = 1 - \left(1 - e^{-\frac{3}{5}}\right) = e^{-\frac{3}{5}} \approx .549$

3. Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

   Solution: The bulb is memoryless, so $\Pr[X - 30 > 30 \,|\, X > 30] = \Pr[X > 30] = e^{-\frac{3}{5}} \approx .549$

Normal Distribution

1. The average jump of a certain frog is 3 inches. However, because of the wind, the frog does not always go exactly 3 inches. A zoologist tells you that the distance the frog travels is normally distributed with mean 3 and variance $\frac{1}{4}$. Recall that given a random variable X distributed normally with mean μ and variance σ^2, the random variable $Z = \frac{X-\mu}{\sigma}$ is distributed normally with mean 0 and variance 1. Observe that this implies that $\Pr[Z \le z] = \Pr[X \le \sigma z + \mu]$. Let us define the function $\Phi(z)$ as the CDF of the standard normal, i.e., $\Phi(z) = \Pr[Z \le z]$ where $Z = N(0,1)$. For the following problems, write your answer in terms of $\Phi(z)$. Then use the table on the back of this sheet to get a numerical answer.

   (a) What is the probability that the frog jumps more than 4 inches?

   Solution: $\Pr[X > 4] = \Pr[X - 3 > 1] = \Pr[\frac{X-3}{\frac{1}{2}} > 2] = 1 - \Pr[Z < 2] = 1 - \Phi(2) = 1 - .9772 = .0228$

   (b) What is the probability that the distance the frog jumps is between 2 and 4 inches?

   Solution: $\Pr[2 < X < 4] = 1 - (\Pr[X > 4] + \Pr[X < 2])$. From the symmetry of the normal curve, $\Pr[X > 4] = \Pr[X < 2]$. So $\Pr[2 < X < 4] = 1 - 2\Pr[X > 4] = 1 - 2 \cdot (1 - \Phi(2)) = .9544$

2. The Cal basketball team plays 100 independent games, each of which they have probability 0.8 of winning. Use the Central Limit Theorem and the table on the back of this sheet to estimate the probability that they win at least 90 games.

   Solution: X, the number of games Cal wins in total, is distributed as $Bin(n = 100, p = .8)$. $\mu = np = 100 \cdot .8 = 80$. $\sigma^2 = np(1-p) = 100 \cdot .8 \cdot .2 = 16$. $\Pr[X > 90] = \Pr[X - 80 > 10] = \Pr[\frac{X-80}{4} > \frac{5}{2}] \approx 1 - \Phi(\frac{5}{2}) \approx 1 - .9938 = .0062$.

Here is a table of approximations of the CDF of the standard normal distribution. To look up the $\Pr\left[N(0,1) \leq x\right]$, you look up the row of the first two digits of x and then the column of the third digit. For example, to look up $x = 1.34$, first find the row labeled 1.3 and then the column 0.04. The result is the entry in that cell, 0.9099.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990