Infinity and Uncountability

1. For each of the following functions from $\mathbb{R}$ to $\mathbb{R}$, determine whether it is an injection, surjection, bijection, none of the above, or more than one of the above.

   (a) $f(x) = 2^x$
   Solution: Injection. $f(x)$ cannot be non-positive values.

   (b) $f(x) = x^2$
   Solution: None of the above. Not an injection because every non-zero $f(x)$ occurs twice. Not a surjection because $f(x)$ cannot be negative values.

   (c) $f(x) = 2x + 1$
   Solution: More than one of the above (bijection, injection, surjection). There is exactly one x that maps to any given value, namely $f^{-1}(y) = \frac{y-1}{2}$.

2. For each of the following sets, decide whether it is countable or uncountable, and justify your answer.

   (a) The set of all prime numbers
   Solution: Countable. There exists a bijection from primes to a subset of natural numbers, such as primes to primes.

   (b) The set of all finite sequences of integers
   Solution: Countable. Consider the following algorithm that enumerates these sequences. For $n = 0$ to ∞, list all sequences (that haven't already been listed) of length $\leq n$ whose entries have absolute value $\leq n$. Each iteration of the loop is finite, since for a given n, there are finitely many sequences of length $\leq n$ with all entries having absolute value $\leq n$. This means that the algorithm is guaranteed not to loop endlessly on any iteration of n. Since any finite sequence of integers has some n such that it will be listed by this algorithm, this algorithm is guaranteed to list every possiblitiy (given infinite time). Thus, this enumeration implicitly creates a bijection onto the natural numbers, and the set of all finite sequences of integers is countable.

   (c) The set of all real numbers in the range $[0, 0.1]$
   Solution: Uncountable. By diagonalization, there are more real numbers in any nonzero range than there are natural numbers. Note that if real numbers in this range were countable, then by simply adding a constant, real numbers in the ranges $[0.1, 0.2], [0.2, 0.3], \ldots, [0.9, 1.0]$ would all be countable. This would mean that real numbers in the range $[0, 1]$ would be countable, which contradicts what has been shown in lecture.

   (d) The set of all real numbers that are roots of polynomials with natural number coefficients
   Solution: Uncountable if the zero polynomial is included since it has a root at every real number and real numbers are uncountable. Countable if the zero polynomial is excluded. In note 18, we proved that the set of polynomials with natural number coefficients is countable. Each nonzero polynomial with natural number coefficients has finitely many roots. This means that we can

use the following algorithm to enumerate all roots: For each polynomial, for each root the polynomial has, if the root has not been output, output the root. Since each polynomial has finitely many roots, the inner loop will always terminate. Since these polynomials are countable, this algorithm is guaranteed to output the roots of each of these polynomials (given enough time). This means that the algorithm is guaranteed to output every root. By enumerating these roots, we have an implicit bijection to the natural numbers, so the set of real numbers that are roots of polynomials with natural number coefficients is countable.

Alternatively, consider the following.

Countable if the zero polynomial is excluded. Consider polynomials with natural number coefficients $p_i(x)$ of degree d. The set of polynomials of degree d with can be written as $\{p_1(x), p_2(x), ...\}$. Each polynomial with natural number coefficients has a finite number of roots. Thus, the roots $r_{i,j}$ of these polynomials can be written as $R_d = \{r_{1,1}, r_{1,2}, ..., r_{1,d}, r_{2,1}, ..., r_{2,d}, ...\}$. This list will enumerate each possible root, so it is countable. This means that R, the set of roots of all polynomials with natural number coefficients, is $\cup_{i=0}^{\infty} R_i$. Now consider two lemmas.

Lemma 1: If A is countable and B is countable, then $A \cup B$ is countable.

Proof: Direct proof. Since A is countable, there exists a bijection from A to a subset of $\mathbb{N}$. Since B is countable, there exists a bijection from B to a subset of $\mathbb{N}$. Consider the bijection from $\mathbb{N}$ to nonnegative even numbers. Using that bijection and the bijection from A to $\mathbb{N}$, there exists a bijection from A to a subset of nonnegative even numbers. Using the bijection from $\mathbb{N}$ to positive odd numbers and the bijection from B to $\mathbb{N}$, there exists a bijection from B to a subset of positive odd numbers. This means that $A \cup B$ has a bijection onto a subset of the union of nonnegative even numbers and positive odd numbers, which is just $\mathbb{N}$. This means that $A \cup B$ is countable.

Lemma 2: If $C = \{C_1, C_2, ...\}$ is a countable set of countable sets, then $\cup_i C_i$ is countable.

Proof: Direct proof. By the first lemma, the union of any two C_i is countable. This means that the union of any three C_i is also countable. Continuing this process a countable number of times always results in a countable set (this can be proved more rigorously with formal induction), and thus the union of all C_i must be countable.

By lemma 2, our set R of roots of all polynomials with natural number coefficents is countable.

3. Prove that if A is uncountable and B is a countable subset of A, then $A - B$ is uncountable.

   Solution: We will use proof by contradiction. Suppose for contradiction that A is uncountable, B is a countable subset of A, and $A - B$ is countable. This means that there exists a bijection from $A - B$ to a subset of $\mathbb{N}$. Since B is countable, this means that there exists a bijection from B to a subset of $\mathbb{N}$. Consider a bijection from $A - B$ to (possibly a subset of) nonnegative even numbers and a bijection from B to (possibly a subset of) positive odd numbers. This means that "combining" the two bijections gives us a bijection from A to (possibly a subset of) $\mathbb{N}$. However, if there exists a bijection from A to a subset of $\mathbb{N}$, then A is countable. This is a contradiction on the assumption that A is uncountable, and thus if A is uncountable and B is a countable subset of A, then $A - B$ is uncountable.