

Sample Spaces, Basic Probability and Conditional Probabilities

Suppose that one of your friends is starting a Combinatorics Club, and n people have signed up to join. The club is still in its formative stages, and you need to hold elections in order to choose the club's leadership. Your friend decided that the club should be governed as follows: there should be one club president, plus k cabinet members who are all equal in power, and they assist the president.

1. At first, your friend wants to nominate $k + 1$ people ahead of time, and one of these people will be randomly selected to be president; the rest will be cabinet members. Let the nominees be p_1, p_2, \dots, p_{k+1} .
 - (a) What is the *sample space* for the leadership groups?
 - (b) What is one *outcome* in this sample space?
 - (c) Give one example of an *event* in this sample space.
 - (d) Assuming you are one of the nominees, what is the probability that you will be part of the leadership (president or cabinet member)?
 - (e) Assuming you are one of the nominees, what is the probability that you will be president?
 - (f) Assuming you are one of the nominees, what is the probability that you will be a cabinet member?
 - (g) Now, assuming instead that the nominees were initially chosen uniformly at random, what is the probability you will be president?
2. Your friend changes her mind, and decides that it is best to allow each of the n people signed up to run for president, and to allow each of the candidates to choose a vice president.

Alice, Bob, Chuck, and Delilah all sign up as candidates, and each has a list of people they equally like. They will select their vice president randomly from this list.

Candidate	Vice President Options
Alice	Emily, Fred, Gary, Hilda
Bob	Alice, Emily, Gary, Jordan, You
Chuck	Alice, Bob, You
Delilah	Chuck, Gary, Iggy, Jordan

Notice that these lists overlap (for example, you can be either Bob or Chuck's VP).

- (a) What is the *sample space* for the leadership groups?
- (b) What is one *outcome* in this sample space?
- (c) Give one example of an *event* in this sample space.
- (d) Assuming the winning candidate is chosen uniformly at random, what is the probability that Gary will be part of the leadership (president or VP)?

- (e) Assuming Emily becomes vice president, what is the probability that Gary is part of club leadership?
 - (f) Assuming that Alice has a %30 chance of winning, Bob has a %40 chance of winning, Chuck has a % 20 chance of winning and Delilah has a %10 chance of winning, what is the probability that You are part of the club leadership?
 - (g) Suppose that due to a scandal, Delilah drops out of the election. Conditioning on the fact that Delilah does not win, and given that Alice originally had a %30 chance of winning, Bob had a %40 chance of winning, and Chuck had a %20 chance of winning, what is the probability that Alice is part of the club leadership?
3. Your friend decides that Alice, Bob, Chuck and Delilah are not really fit to lead Combinatorics Club. In a stroke of luck, Delilah drops out of the race due to a scandal, Chuck drops out because he decides that being president would be too much on top of his courseload, and Bob drops out to spend more time with his family. So it seems like Alice will win by default.
- To avoid this, your friend enters you into the election as a candidate, and she will rig the election so that you are more likely to win. When the votes come in, starting from day 0, every day she will sample a vote by drawing it uniformly at random from a hat. If the vote is for you she will declare you the victor. If the vote is for Alice, then if it is an even-numbered day she will return it to the hat, but if it is an odd-numbered day she will declare that Alice won.
- (a) Assuming there were an equal number of votes for you and for Alice, what is the probability that you win the election?
 - (b) Assuming that Alice had %70 of the vote, what is the probability that you win the election?
 - (c) **Extra:** What is the minimum proportion of the vote that Alice needs in order to win the election with probability at least $\frac{1}{2}$?
4. **Challenge:** Recall the Monty Hall Problem from lecture: A contestant is participating in a game show, in which there are 3 doors. Behind two of the doors are goats, behind one of the doors is a prize; if the contestant chooses the door with the prize behind it, they can keep the prize.
- When the contestant selects a door, the game show host opens one of the other two doors to reveal a goat. Then, the contestant has the option of changing doors.
- What is the probability that the contestant gets the prize if they stay at the current door? What is the probability that the contestant gets the prize if they switch to the other door?