CS 70 Discrete Mathematics and Probability Theory
Fall 2013 Week 1 Discussion

Induction

Note: you should not worry at all if you weren't able to get through these problems in section, or if you find some of the problems (especially 3 and 4) very difficult. Exams will not contain questions which require as much ingenuity as questions 3 or 4.

1. Prove that for all $n \in \mathbb{N}$, $5 \mid 6^n - 1$.

   Solution: Let $a_n = 6^n - 1$. We prove $5 \mid a_n$ by induction on n.

   Base case: $a_0 = 0$, which is a multiple of 5.

   Inductive hypothesis: Suppose $5 \mid a_n$.

   Inductive step: Note that $a_{n+1} - a_n = 6^{n+1} - 6^n = 6^n(6-1)$. By the inductive hypothesis, we can find k such that $a_n = 5k$. Then $a_{n+1} = 5k + 6^n(6-1) = 5(k+6^n)$, so $5 \mid a_n$..

2. In a certain nation, each president wants to show up their predecessors, and so makes *more* speeches than all previous presidents combined. Assuming the first president makes at least one speech, prove that the n^{th} president must make at least 2^{n-1} speeches for each $n \in \mathbb{N}$.

   Solution: Let a_n be the number of speeches the n^{th} president makes. We will prove that $a_n \geq 2^{n-1}$ by strong induction on n.

   Base case: By the problem statement, $a_1 \geq 1 = 2^{1-1}$.

   Inductive hypothesis: Suppose for each $k \leq n$, $a_k \geq 2^{k-1}$.

   Inductive step: By the problem statement, $a_{n+1} > a_1 + a_2 + \ldots + a_n$. Thus $a_{n+1} \geq 1 + a_1 + a_2 + \ldots + a_n$. By the inductive hypothesis, $a_{n+1} \geq 1 + 1 + 2 + \ldots + 2^{n-1}$. We can then verify by a second induction that this sum evaluates to 2^n: for the base case note that $1 + 1 = 2^1$, and then for the indutive step note that $2^{n-1} + 2^{n-1} = 2^n$.

3. Suppose that you are interested in retrieving an object located in the middle of the desert, n kilometers away. Your car can carry enough fuel to travel 3 kilometers, and you have an unlimited supply of spare fuel tanks which you can use to leave deposits of fuel throughout the desert. Your starting point has as much fuel as you want.

   (a) Show that it is possible to retrieve the item and return it to your starting location by driving at most 3^n kilometers.

   Solution: Let a unit of gas be the amount of gas necessary to travel 1 kilometer. Let p_n be the point n kilometers away from our starting location (the starting location itself is p_0). We will prove by induction that there is a strategy which starts and ends with the car at p_0 and which deposits 1 unit of gas at p_n using only 3^n units in total (including the unit left behind).

   This is strictly stronger than the desired statement, because we could simply take the desired item back with us on our last return from p_n. Moreover, if we only use 3^n units of gas then we can only travel $3^n - 1$ kilometers (since we left one unit behind).

Base case: If $n = 0$, we can perform the task trivially just by taking our $3^0 = 1$ unit of gas and leaving it at our starting location.

Inductive hypothesis: Assume that we have a procedure $A(n)$ which uses 3^n units of gas, which starts and ends with the car at p_0, and which leaves 1 unit of gas at p_n.

Inductive step: In order to leave 1 unit of gas at p_{n+1}, we use the procedure $A(n)$ three times, to leave 3 units of gas at p_n. However, the third time we run A we make a slight modification.

Consider the last time the car is at p_n. At that time, there are 3 units of gas at p_n. We modify $A(n)$ by storing the car's current contents at p_n and then refilling the car with those 3 units of gas. We then use 1 of these units of gas to travel to p_{n+1}, we leave 1 of these units at p_{n+1}, and we use 1 unit to return to p_n. At this point we restock the car using the gas that we left at p_n, and then finish running $A(n)$.

This protocol requires running $A(n)$ three times, but uses no additional gas (recall that we counted the 1 unit left behind in the 3^n units of gas that $A(n)$ uses). By induction, this protocol uses $3 * 3^n = 3^{n+1}$ units of gas. And it clearly leaves 1 unit of gas at p_{n+1}, as desired.

Thus we can take all of the gas currently in the car and set it aside. We can then put those 3 units of gas into our car, use one unit to travel one kilometer further, leave one unit at $n+1$ kilometers, and use one unit to return to n

(b) For the adventurous: what is the minimal number of miles it is possible to travel in order to retrieve the item?

Solution sketch: The key part of the analysis above was starting with 3 units of gas at p_n and then ending with 1 unit of gas at p_{n+1}. However, we could have made a different tradeoff. For example, we could have started with 3 units of gas at p_n and travelled only $\frac{1}{2}$ of a kilometer, to end up with 2 units of gas at $p_{n+\frac{1}{2}}$. In general, we could move $3-2\varepsilon$ units of gas to $p_{n+\varepsilon}$.

By the exact same analysis as in the last section, this change allows us to move the gas using only $(1-\frac{2}{3}\varepsilon)^{-n/\varepsilon}$ kilometers. For every small values of ε, this approaches $\exp\left(\frac{2}{3}n\right)$.

Proving this is (roughly) optimal involves much harder techniques, which you aren't expected to know (yet). Briefly: define the *value* of gasoline at p_x to be $\exp\left(\frac{2}{3}x\right)$ per unit. We can prove that driving the car can never increase the total value of our gasoline—the increase in the gasoline's value from carrying it away from the starting point is more than offset by the cost of the gasoline that is consumed. Ending up with a unit of gasoline at p_n means you end up with a total value of at least $\exp\left(\frac{2}{3}n\right)$. Since driving the car doesn't create value, you must have started with $\exp\left(\frac{2}{3}n\right)$ value, i.e. you must have started with that much gas at the starting location.

(And to ever reach p_n requires getting a unit of gas to p_{n-1}. So this analysis is correct to within a factor of $\exp\left(\frac{2}{3}\right)$. I don't know exactly how much gas you need...)

4. Suppose you have a calculator on which the only working keys are $3, 6, 9, (,), +, *, -$, and the decimal point. Find a way to enter 1.7 into this calculator, or show that it is impossible.

Solution: We apply the well-ordering principle, together with strengthening the hypothesis. First we observe that most of the expressions we can make seem to be "almost" a multiple of 3, but with a decimal point added in. Experimenting and formalizing this observation, it seems like we can only make numbers of the form $\frac{3k}{10^\ell}$. Say that a number is of type P if it is of the form.

First, we'll prove that 1.7 isn't of type P. Suppose that $1.7 = \frac{3k}{10^\ell}$. First notice that $\ell > 0$, since $3k$ is an integer and 1.7 is not. But if 1.7 is of the form $\frac{3k}{10^\ell}$ with $\ell > 0$, then $3k = 17 * 10^{\ell-1}$. But 3 does not divide $10^{\ell-1}$ nor 17, and 3 divides a product if and only if it divides one of the factors. So this is a contradiction, and 1.7 isn't of type P.

So we will be done if we can show that every number we can produce is of the form $\frac{3k}{10^\ell}$. Say that a number is of type P if it has this form. We apply the well-ordering principle, and consider the smallest expression which we can enter which *isn't* of type P.

This expresion is of one of four forms. We will show that any possibility leads to a contradiction:

(a) If the expression of the form $a+b$ for smaller expressions, then by hypothesis each of these expressions is of type P, and the sum of two numbers of type P is also of type P.

$$\frac{3k}{10^\ell} + \frac{3k'}{10^{\ell'}} = \frac{3\left(k10^{\ell'} + k'10^{\ell}\right)}{10^{\ell+\ell'}}.$$

(b) Similarly if the expression is of the form $a-b$ or $a*b$.

(c) If the expresison is simply a number, consisting of the digits $3,6,9$ and the decimal point, then it is of the form

$$x = \sum_i a_i * 10^i$$

with each a_i equal to one of $3,6$, or 9. But each summand is manifestly of type P, and as we saw in part (a), a sum of terms of type P is also of type P.

So every number we can produce is of type P, but 1.7 isn't, and so we cannot produce 1.7.

5. Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2} = F_{i+1} + F_i$ and $F_0 = 0$, $F_1 = 1$. Prove that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$$

Solution: We proceed by induction on n.

Base case: $\sum_{i=0}^{0} F_i^2 = F_0^2 = 0 = F_0 F_1$

Inductive hypothesis: Suppose $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$

Inductive step: We have

$$\begin{aligned}\sum_{i=0}^{n+1} F_i^2 &= F_{n+1}^2 + \sum_{i=0}^{n} F_i^2 \\ &= F_{n+1}^2 + F_n F_{n+1} \\ &= F_{n+1}\left(F_n + F_{n+1}\right) \\ &= F_{n+1} F_{n+2}\end{aligned}$$

where the second equality is the inductive hypothesis and the last equality is the definition of the Fibonacci numbers.