CS 70 Discrete Mathematics and Probability Theory
Fall 2013 Week 2 Discussion

Induction and Stable Marriage

1. Suppose that I start with 0 written on a piece of paper. Each minute, I choose a digit written on the paper and erase it. If it was 0 I replace it with 010. If it was 1 I replace it with 1001. Prove that no matter which digits I choose and no matter how long the process continues, I never end up with two 1's in a row.

   Solution: Conider the first step at which I obtain two 1's in a row. Note that this can't be the first step, since 0 doesn't have two 1's in a row. Let x be the digit I erased, and let y be the new expression I substituted for x. y doesn't contain two consecutive 1's, so at least one of the new 1's must lie outside of y. But before erasing x there was no pair of consecutive 1's, so at least one of the new 1's must lie within y.

   If x was a 0 then this is impossible, because the first and last digits of y are 0. If x was a 1 then this is also impossible, because in this case the digits before and after x must have been 0, since there was no pair of consecutive 1's.

2. Suppose that in a certain city there are N intersections, some pairs of which are connected by bidirectional roads. The roads are such that it is possible to travel from one intersection to any other intersection over the roads. Sometimes civil engineers will close a road for construction (which they will later reopen). However, whenever they close a road, it will be part of some *loop*: a sequence of connected roads $R_1, R_2, \dots, R_k$ which start and end at the same location, in which each road appears at most once. Prove that it is always possible to get from any intersection to any other intersection.

   Solution: Suppose for a contradiction that the claim is false, and consider the first start of construction at which it becomes impossible to move between two intersections (note that it is possible to move between any two intersections at the start of the process, by assumption). Let X_i be the road under construction, and let $X_1, X_2, \dots, X_k$ be a path between A and B before construction began, such that there is no path between A and B after construction begins. By assumption X_i is part of some tour $R_1, R_2, \dots, R_m$. For convenience, assume $X_i = R_1$ (otherwise renumber the R's so that this is true).

   Then $X_1, X_2, \dots, X_{i-1}, R_2, R_3, \dots, R_m, X_{i+1}, X_{i+2}, \dots, X_k$ is a path from A to B, contradicting the claim that there is no path from A to B.

3. Run the Stable Marriage Algorithm on the following preference table:

Man	Women			
1	B	A	C	D
2	D	C	B	A
3	A	C	B	D
4	A	D	B	C

Woman	Men			
A	2	1	3	4
B	3	4	2	1
C	1	3	4	2
D	1	4	2	3

   Solution: In the first round, 1 proposes to B, 2 proposes to D, and 3 and 4 propose to A. B and D accept, while A accepts 3 and rejects 4. In the second round, 4 proposes to D, and D accepts 4 while rejecting 2. In the third round, 2 proposes to C, who accepts.

4. Suppose Alice, Bob, and Charlie are three people participating in the propose-and-reject algorithm. Suppose Alice rejects bob's proposal and ends up matched with Charlie. Prove that if Alice rejects Bob's proposal and ends up matched with Charlie, then Alice prefers Charlie to Bob.

   Solution: Suppose Alice doesn't prefer Charlie to Bob, and consider the first round of the algorithm at which Alice accepts someone (say Dave) whom she *doesn't* prefer to Bob. Since Dave was the first person she accepted whom she didn't prefer to Bob, Alice either rejected Bob himself or rejected someone else (Ernest), whom she preferred to Bob. But in either case she wouldn't prefer Dave to the person whom she rejected, which contradicts the definition of the propose-and-reject algorithm.

5. Prove that the matching produced by the propose-and-reject algorithm is *stable*: there is no pair of people (Alice, Bob), such that Alice prefers Bob to her partner under the matching *and* Bob prefers Alice to his partner under the matching.

   (Hint: consider separately the cases where Alice and Bob are paired, where Alice rejected Bob, and where Bob never proposed to Alice.)

   Solution: By construction of the propose-and-reject algorithm, either Alice and Bob are matched, Bob never proposes to Alice, or Alice rejects Bob. If Alice and Bob are matched, then Alice can't prefer Bob to her match (since Bob *is* her match). If Alice rejects Bob, then by the previous problem we know that Alice ends up with someone she prefers to Bob. If Bob never proposes to Alice, then Bob only ever proposes to people he prefers to Alice, and so he ends up with someone he prefers to Alice.

   In every case, either Alice doesn't prefer Bob to her match, or Bob doesn't prefer Alice to his match.

6. Based on the preference tables from problem 3, indicate whether each of the following pairings is stable. For each person, which of the stable matchings do they most prefer? Which do they least prefer?

   (a) $\{(1,B),(2,C),(3,A),(4,D)\}$

   (b) $\{(1,D),(2,B),(3,C),(4,A)\}$

   (c) $\{(1,C),(2,A),(3,B),(4,D)\}$

   (d) $\{(1,A),(2,B),(3,C),(4,D)\}$

   Solution: A, B, D are stable. B is not stable, because 1 would prefer be with C than D, and C would prefer be with 1 than 3, so $(1,C)$ is a "rogue couple."

   Amongst the stable matchings, every man prefers A and disprefers C, while every woman prefers C and disprefers A.