CS 70 Discrete Mathematics and Probability Theory
Fall 2012 Vazirani Week 6 Discussion

Introduction to Graphs

Note: you aren't expected to complete even all of the non-challenge problems. Extra problems are included to help with practice.

1. Give the necessary and sufficient conditions for an undirected connected graph to have an Eulerian walk.

   Solution.

   (a) **Prove: If an undirected graph G has an Eulerian walk W, the graph can have at most two odd degree vertices.**
   For a vertex v, let $d(v)$ be the degree of v and let $n_W(v)$ be the number of edges on W incident to v. Since W is Eulerian, $n_W(v) = d(v)$. Also observe that $n_W(v)$ must be even for all vertices other than the start and end vertices of the W. This is because each time W enters an intermediate vertex, it must exit it so W uses up two edges incident to it. Therefore, there can be at most two odd degree vertices (the start and end vertex).

   (b) **Prove: If a connected graph has at most two odd degree vertices, it has an Eulerian walk.**
   We learned in lecture that there must be an even number of odd degree vertices, since the sum of the degrees of all vertices must be even. Therefore, there are either 0 odd degree vertices or 2 odd degree vertices. If there are 0, we have an Eulerian tour by Euler's theorem. If there are 2 odd degree vertices, u and v, we can construct an Eulerian walk.

   We will start our walk T from vertex u, never repeating edges. Let's say we get stuck at vertex w. If $w = u$, then $n_T(w)$ is even, since we started at u. But since $d(u)$ is odd, there must exist at least one unused edge incident to u which we can use to continue. If $w \neq u$, then $n_T(w)$ is odd; we have entered w but not exited. If w is even degree, there must exist at least one unused edge incident to w which we can use to exit. Therefore, w must be odd degree, so w must be v.

   We've created a walk T, but it is not necessarily Eulerian. If we remove T from the graph, the remaining graph has even degree. This is because we are subtracting $n_T(w)$ from the degree of each vertex w. For the even degree vertices (vertices other than u and v), we are modifying the degree by subtracting an even number, resulting in an even degree. For u and v we are subtracting odd numbers from their degrees, resulting in even degrees. Now we can construct Eulerian tours in the remaining graph. These tours might be disconnected, but they can all be connected to T. This is due to Claim 3 from note 7 (we can generalize Claim 3 so that A is a trail and not a tour- the proof remains the same). Finally, we splice together T with all the Eulerian tours to create the final Eulerian trail.

2. A *Hamiltonian path* is a path that visits each vertex exactly once. A *tournament graph* is a directed graph such that for all vertices u, v in the graph, either $(u, v) \in E$ or $(v, u) \in E$. Show that a tournament

graph has a Hamiltonian path.

Let's proceed by induction on the number of vertices, n. For the base case, consider $n = 2$. Then we just have one edge between the two vertices which constitutes a Hamiltonian path. For the inductive hypothesis, assume a tournament graph with $n-1$ vertices has a Hamiltonian path. Now if G has n vertices, pick an arbitrary vertex w and remove w and all edges incident to w to create graph G'. G' is still a tournament graph, so it has a Hamiltonian path $(v_1, v_2), \ldots, (v_{n-2}, v_{n-1})$. If $(v_{n-1}, w) \in E$, we can add this edge to the end of the Hamiltonian path in G' to obtain a Hamiltonian path in G.

If not, find the smallest i such that $(w, v_i) \in E$. Note that such an i exists, since $(w, v_{n-1}) \in E$. The Hamiltonian path in G is then $(v_1, v_2), \ldots, (v_{i-2}, v_{i-1}), (v_{i-1}, w), (w, v_i), (v_i, v_{i+1}), \ldots, (v_{n-2}, v_{n-1})$.

3. **Challenge problem**: In lecture, you learned that an undirected graph $G = (V, E)$ has an Eulerian tour if and only if the graph is connected (except for isolated vertices) and even degree. Prove the following alternate characterization of Eulerian graphs: A connected graph G has an Eulerian tour if and only if its edge set can be decomposed into disjoint cycles (two cycles are disjoint if they share no edges). Hint: try using induction on the number of edges in the graph.

   (a) **Prove: If G has an Eulerian tour, its edge set can be decomposed into cycles** We can prove this using induction on the number of edges (assuming the number of vertices is fixed). The base case is $|E| = 0$, and for the inductive hypothesis we assume that if G has an Eulerian tour and has at most $m-1$ edges, its edge set can be decomposed into cycles. Now we'd like to prove the above statement for a graph G with m edges.

   Let the Eulerian tour of G be $(v_1, v_2), \ldots, (v_n, v_1)$. A cycle within the above Eulerian tour would be of the form $(v_i, v_{i+1}), \ldots, (v_{i+k-1}, v_{i+k})$, where $v_i, \ldots, v_{i+k-1}$ are distinct and $v_i = v_{i+k}$. Consider the smallest such cycle C (minimize k). Remove this cycle from G. The resulting graph G' cannot have more than $m-1$ edges, and has Eulerian tour $(v_1, v_2), \ldots, (v_{i-1}, v_i), (v_{i+k}, v_{i+k+1}), \ldots, (v_n, v_1)$. By the inductive hypothesis, the edge set of G' can be decomposed into disjoint cycles. The edge set of G can therefore be decomposed into disjoint cycles consisting of all cycles from G' and the cycle C.

   (b) **Prove: If G is connected and its edge set can be decomposed into cycles, it has an Eulerian tour** We will instead prove that G has even degree, which implies that G has an Eulerian tour. If G's edge set can be decomposed into cycles, consider the number of cycles n incident at vertex v. The degree of v is therefore $2n$; the cycle starts at v and returns to v. Therefore, G is an even degree graph.