Counting

Counting Committees:

1. **Let's say you have a company with n employees, and you would like to form a committee of k employees. Each employee in the committee is designated to perform one of k different tasks.**

   (a) **If each employee can only perform one task at a time, how committee choices do you have?** This is sampling without replacement and with ordering, so we have $n \cdot (n-1) \cdots (n-(k-1))$ committee choices by the first rule of counting.

   (b) **If each employee can perform up to k tasks at a time, how many choices do you have?** This is sampling with replacement and with ordering, so we have n^k committee choices again by the first rule of counting.

2. **Now assume you don't have designated tasks; you simply want to form a committee of k employees who will work together on one task. How many different committees are possible?**

   (a) **If the work is divided evenly between the k chosen employees (so you actually need k committee members), how many different committees are possible?** We are now sampling without replacement and without ordering. By the second rule of counting, we have $\binom{n}{k}$ possible committees.

   (b) **Challenge: What if instead of needing k employees, you want k hours of committee work done, and employees sign up to be part of the committee by signing up for some number of 1-hour work increments (notice it's possible for only one person to complete all k hours of the committee work). In this setting, how many ways can you choose the committee?** This is the same as the fruit picking example in the notes - we are sampling with replacement and without ordering. To draw an analogy between this problem and fruit picking, let k be the number of fruits in the salad and n is the number of different types of fruit available. The number of committee choices possible is then $\binom{n+k-1}{k}$.

Counting and Probability:

1. **Go back to the example above in which you have n employees, and you want to form a committee of size k in which each committee member does a different job. Say each employee can only perform one task at a time.**

   (a) **Say that one of the jobs is being committee chair. Suppose you have one most trusted employee, X. How many committee choices do you have if you always designate X as the chair?** Now we are fixing one of our choices, say the first, to be employee X. Then for our second choice we have $n-1$ options, and so on. Our total number of committee choices is then $(n-1) \cdot (n-2) \cdots (n-(k-1))$.

(b) If you choose a random committee, what is the probability that X is the committee chair? This is just the number computed in the previous part divided by the answer in 1(a) above - so we have $\frac{1}{n}$.

2. **Now, say you want a committee with k jobs, and each person can do more than one job.**

   (a) **Say that you always make sure X is the committee chair, and that he also has the job of note-taker. How many committee choices do you have if you always give both of these jobs to X?** We have $k-2$ jobs remaining, and n options for each of the $k-2$ jobs. So we have n^{k-2} possible committee choices.

   (b) **If you choose committees completely at random, what is the probability that X is chair and note-taker of the committee?** This is the answer computed in the previous part divided by the answer computed in 1(b) above: $\frac{n^{k-2}}{n^k} = \frac{1}{n^2}$.

3. **Finally, say you want a committee in which all members do an equal amount of work.**

   (a) **Now, how many committee choices do you have if you always make sure X is part of the committee?** Since employee X is already chosen, we are now choosing a committee of $k-1$ members from $n-1$ employees. We therefore have $\binom{n-1}{k-1}$ options.

   (b) **If you choose a random committee, what is the probability that X is part of the committee?** This would be the answer to the previous part divided by the answer to 2(a), which is $\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$.