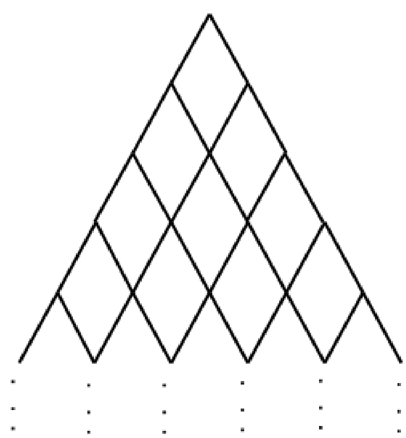
# Probability Theory

Probability theory is one of the great achievements of the 20th century. In statistical physics it provides a technique for understanding the behavior of large numbers of particles, without analyzing their individual behavior. In quantum mechanics, it is an integral part of the fabric of the theory. Probabilistic methods form the backbone of the recent achievements of machine learning, algorithms, and artificial intelligence.

One of the basic themes in probability theory is the emergence of almost deterministic behavior from probabilistic phenomena. For example, if we flip a fair coin 10,000 times, we almost certainly expect between 4900 and 5100 heads. This phenomenon is responsible for the stability of the physical world we live in, whose building blocks - elementary particles, atoms, molecules - individually behave in many ways like coin flips, but in aggregate present a solid front to us, much like the number of heads in the example above.

## Coin Flipping

Let's look at the coin flipping example in more detail. To picture coin flipping, consider the simple apparatus below called the Galton-Watson board.

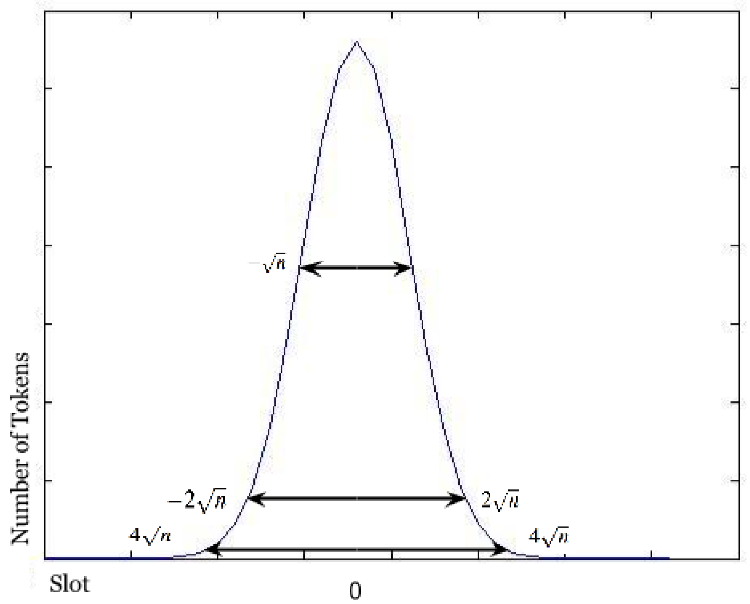


Imagine we drop a token in the slot at the top of the board. As the token falls, at each peg it chooses randomly with 50-50 chance whether to go right or left. If we were to watch a single token drop, it would end up in one of the slots at the bottom of the board, chosen in some random way. What happens if we watch $N$ tokens drop? How many tokens does each of the bottom slots hold?

Before we answer the question, let us observe that the Galton-Watson board is just a way of visualizing a sequence of coin flips. To see this, imagine that each time the token encounters a peg, it flips a coin to decide whether to go left or right. If the coin is heads, the token goes right, and if the coin is tails, the token goes left.

If the height of the Galton-Watson board is $n$, then there are exactly $n$ coin flips as the token makes its way down. Moreover, if we think of heads as +1 and tails as -1, then the slot that the token ends up in is just the sum of the choices at each round, which ranges from $-n$ to $n$. So we can label each slot by a number between $-n$ and $n$ in this way, with the leftmost slot labelled $-n$ and the right most slot is labelled $n$. Notice that the slot number also represents the number of tails subtracted from the number of heads.

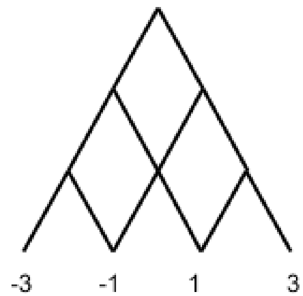
Now let's get back to our question - what happens if we watch $N$ tokens drop? Here is a figure showing the answer for a board of height $n$:




As you can see, when $n = 10,000$, we expect almost all the tokens to fall in slots between -200 and 200. If we go back to the coin flipping setting, we can see that we should expect the number of tails subtracted from the number of heads to be between -200 and 200; equivalently, the number of heads is expected to be between 4900 and 5100.

What is the significance of the range 4900 to 5100? Note that the size of that interval 5100-4900 = 200 is the twice the square root of $n = 10,000$. This is no coincidence. In general if we flip a coin $n$ times, the probability of landing in a slot between $-2\sqrt{n}$ and $2\sqrt{n}$ is at least $\frac{3}{4}$, but the probability of landing outside of slots between $-4\sqrt{n}$ and $4\sqrt{n}$ is at most $\frac{1}{16}$. This rapid decline in probability is evident in the picture above - as you move further away from the center of the picture, the number of tokens in the corresponding slots decreases rapidly.

Why does the probability fall so quickly? To understand this, we can turn back to the Galton-Watson branching process. First consider a smaller example, where $n = 3$:




There are eight total paths from the start slot to one of the final slots. There is only one path leading to the slot labelled $-3$ (the token must choose the left branch at each step). The same applies for the right most slot. However, there are 3 paths to each middle slot (labelled $-1$ and 1).

As $n$ grows, the total number of paths that the token can take increases exponentially. In fact, since there are 2 choices for each coin flip, there are $2 \times 2 \times \cdots \times 2 = 2^n$ choices of paths for the token if the board has height $n$. Even for relatively small values of $n$ like 50 or 100, $2^n$ is already astronomically large. Moreover, as the number of paths grows, the discrepancy between the number of paths leading to each slot is magnified.

For example, for a token to land in either the left most or right most slot, it would have to choose one out of $2^n$ paths! The vast majority of the paths end up at one of the slots near the middle. This discrepancy between the likelihood of landing in a central spot rather than a spot closer to one of the two ends is responsible for the almost deterministic behavior described above; this is why we can say that if we flip a coin 10,000 times, we expect the number of heads to be between 4900 and 5100.

# Bias Estimation

So far we have been considering the case where the coin we flip has equal probability of landing on heads or tails. What happens when it is biased- when the probability of heads is $p$ rather than $\frac{1}{2}$? In this case, if we flip the coin $n = 10,000$ times, we expect to see 7500 heads and 2500 tails. Once again, if we repeat this experiment $N$ times, we expect almost all the experiments to end up with between 7400 and 7600 heads.

This behavior suggests that if we did not know the bias $p$ to begin with, we could simply divide the number of observed number of heads by the total number of heads, say $\hat{p} = 7470/10000$ to get an estimate for $p$.

If we could ensure that such an estimate is accurate, it would be useful in various settings. For example, we can use it in election polling. Let's say we would like to estimate the fraction of the population who supports one of two candidates. Polling a random sample of the population is analogous to flipping a coin with bias $p$, where $p$ is the fraction of the population supporting the candidate in the question. Given the polling results, we can use the above technique to estimate $p$.

Can we ensure that our estimate $\hat{p}$ is close tp $p$? The fact that the actual number of heads we observe is almost certain to be within a very narrow range means that we can be pretty confident that our estimate is very close to the actual bias $p$. This is one of the fundamental principles of statistics.

There are two parameters which tell us how good our estimate $\hat{p}$ is. The first is the error of our estimate, $|p - \hat{p}|$. For example, maybe we would like this error to be smaller than some bound $\varepsilon$. i.e. we are looking for an estimate $\hat{p}$ such that $|p - \hat{p}| \leq \varepsilon$.

The second parameter corresponds to our confidence that our estimate is within an error of $\varepsilon$. This is necessary, because when estimating $p$, we can never be completely sure that our estimate $\hat{p}$ is within $\varepsilon$ of $p$. For example, even if $p = \frac{1}{2}$, it might be the case that all our 10,000 coin flips come up heads. In this case, our estimate $\hat{p}$ would be 1. Of course the chance of this is smaller than anything we could imagine ($\frac{1}{2^{10,000}}$ to be exact), but it is still possible. The confidence parameter addresses this possibility. For example, we might say that we are 98% confident that our estimate $\hat{p}$ is within $\varepsilon = .01$ of the actual bias $p$. We can introduce another parameter $0 \leq \delta \leq 1$ for our confidence. So if we are 98% confident, we will say that $\delta = .02$, so that our confidence is $1-\delta$ as a fraction or $100(1-\delta)$ as a percentage.

Both of these parameters - $\varepsilon$ and the confidence percentage- depend on the number of trials. Another way of saying this is that if we wish to achieve a certain error $\varepsilon$, say .1, and a certain confidence, say 95% or $\delta = .05$, then there is a value for $n$ the number of trials, in this case $n = 500$, that will guarantee that with confidence at least 95% our estimate will be within .1 of the actual bias. More generally, as you will see in later lectures, to obtain error $\varepsilon$ and confidence $\delta$ (where $\delta$ is between 0 and 1, so our confidence percentage is 100(1-$\delta$)), $n = \frac{1}{4\varepsilon^2\delta}$ trials would be sufficient.

# Outline

Over the course of the next few lectures, we will develop the tools needed to understand the questions asked above. We'll start with counting and continue to discrete probability. We will then introduce random variables, which will allow us to formalize previous concepts and answer more difficult questions, such as estimating the bias of a coin.

# Counting

Counting lies at the heart of probability theory. In our motivating example of coin flips, for example, counting the number of paths to each of the final slots in the branching process helped us determine that the behavior of a large number of coin flips is almost deterministic. In this section we introduce simple techniques that will allow us to count the total number of outcomes for a variety of probabilistic experiments (including coin flips).

We start by considering a simple scenario. We pick $k$ elements out of an $n$ element set $S = \{1, 2, \cdots, n\}$ one at a time. We wish to count the number of different ways to do this, taking into account the order in which the elements are picked. For example, when $k = 2$, picking 1 and then 2 is considered a different outcome from picking 2 followed by 1. Another way to ask the question is this: we wish to form an ordered sequence of $k$ distinct elements, where each element is picked from the set $S$. How many different such ordered sequences are there?

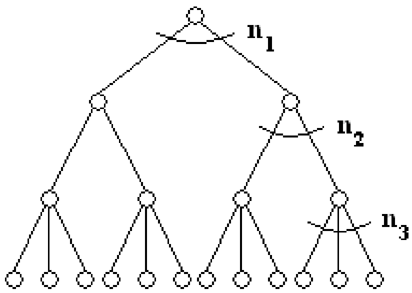
If we were dealing cards, the set would be $S = \{1, \cdots, 52\}$, where each number represents a card in a deck of 52 cards. Picking an element of $S$ in this case refers to dealing one card. Note that once a card is dealt, it is no longer in the deck and so it cannot be dealt again. So the hand of $k$ cards that are dealt consists of $k$ distinct elements from the set $S$.

For the first card, it is easy to see that we have 52 distinct choices. But now the available choices for the second card depend upon what card we picked first. The crucial observation is that regardless of which card we picked first, there are exactly 51 choices for the second card. So the total number of ways of choosing the first two cards is $52 \times 51$. Reasoning in the same way, there are exactly 50 choices for the third card, no matter what our choices for the first two cards. It follows that there are exactly $52 \times 51 \times 50$ sequences of three cards. In general, the number of sequences of $k$ cards is $52 \cdot 51 \cdots (52-(k-1))$.

This is an example of the first rule of counting:

**First Rule of Counting:** If an object can be made by a succession of $k$ choices, where there are $n_1$ ways of making the first choice, and *for every* way of making the first choice there are $n_2$ ways of making the second choice, and *for every* way of making the first and second choice there are $n_3$ ways of making the third choice, and so on up to the $n_k$-th choice, then the total number of distinct objects that can be made in this way is the product $n_1 \cdot n_2 \cdot n_3 \cdots n_k$.

Here is another way of picturing the first rule of counting. Consider the following tree:




It has branching factor $n_1$ at the root, $n_2$ at every node at the second level,..., $n_k$ at every node at the $k$-th level. Each node at level $k+1$ (a leaf node) represents one possible way of making the object by making a succession of $k$ choices. So the number of distinct objects that can be made is equal to the number of leaves in the tree. Moreover, the number of leaves in the tree is the product $n_1 \cdot n_2 \cdot n_3 \cdots n_k$. For example, if $n_1 = 2$, $n_2 = 2$, and $n_3 = 3$, then there are 12 leaves (i.e., outcomes).

# Counting Sets

Consider a slightly different question. We would like to pick $k$ distinct elements of $S = \{1, 2, \cdots, n\}$ (i.e. without repetition), but we do not care about the order in which we picked the $k$ elements. For example, picking elements $1, \ldots, k$ is considered the same outcome as picking elements $2, \ldots, k$ and picking 1 as the last ($k^{th}$ element). Now how many ways are there to choose these elements?

When dealing a hand of cards, say a poker hand, it is often more natural to count the number of distinct hands (i.e., the set of 5 cards dealt in the hand), rather than the order in which they were dealt. As we've seen in the section above, if we are considering order, there are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!}$ outcomes. But how many distinct hands of 5 cards are there? Here is another way of asking the question: each such 5 card hand is just a subset of $S$ of cardinality 5. So we are asking how many 5 element subsets of $S$ are there?

Here is a clever trick for counting the number of distinct subsets of $S$ with exactly 5 elements. Create a bin corresponding to each such 5 element subset. Now take all the sequences of 5 cards and distribute them into these bins in the natural way. Each sequence gets placed in the bin corresponding to the set of 5 elements in the sequence. Thus if the sequence is $(2,1,3,5,4)$, then it is placed in the bin labeled $\{1,2,3,4,5\}$. How many sequences are placed in each bin? The answer is exactly 5!, since there are exactly 5! different ways to order 5 cards.

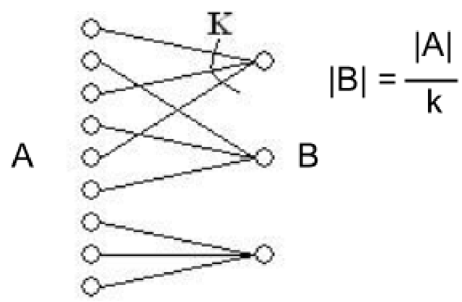
Recall that our goal was to compute the number of 5 element subsets, which now corresponds to the number of bins. We know that there are $\frac{52!}{47!}$ 5 card sequences, and there are 5! sequences placed in each bin. The total number of bins is therefore $\frac{52!}{47!5!}$.

This quantity $\frac{n!}{(n-k)!k!}$ is used so often that there is special notation for it: $\binom{n}{k}$, pronounced *n choose k*. This is the number of ways of picking $k$ distinct elements from $S$, where the order of placement does not matter. Equivalently, it's the number of ways of choosing $k$ objects out of a total of $n$ objects, where the order of the choices does not matter.

The trick we used above is actually our second rule of counting:

**Second Rule of Counting:** Assume an object is made by a succession of choices, and the order in which the choices is made does not matter. Let $A$ be the set of ordered objects and let $B$ be the set of unordered objects. If there exists a $k$ to 1 function $f$ from $A$ to $B$, we can count the number of ordered objects (pretending that the order matters) and divide by $k$ (the number of ordered objects per unordered objects) to obtain $|B|$, the number of unordered objects.

Note that we are assuming the number of ordered objects is the same for every unordered object; the rule cannot be applied otherwise. Here is another way of picturing the second rule of counting:




The function $f$ simply places the ordered outcomes into bins corresponding to the unordered outcomes. In our poker hand example, $f$ will map 5! elements in the domain of the function (the set of ordered 5 card outcomes) to one element in the range (the set of 5 element subsets of $S$). The number of elements in the range of the function is therefore $\frac{52!}{47!5!}$.

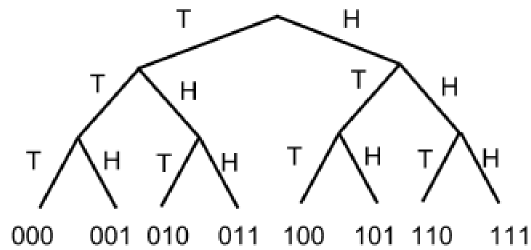
# Sampling with Replacement

Sometimes we wish to consider a different scenario where we are still picking $k$ elements out of an $n$ element set $S = \{1, 2, \cdots, n\}$ one at a time (order matters). The difference is that after we pick an element for our sequence, we throw it back into $S$ so we can pick it again. How many such sequences of $k$ elements can we obtain? We can use the first rule of counting. Since we have $n$ choices in each trial, $n_1 = n_2 = \ldots = n_k = n$. Then we have a grand total of $n^k$ sequences.

This type of sampling is called *sampling with replacement*; multiple trials can have the same outcome. Card dealing is a type of *sampling without replacement*, since two trials cannot both have the same outcome (one card cannot be dealt twice).

## Coin Flipping

Let us return to coin flipping. How many different outcomes are there if we flip a coin $k$ times? By outcome, we mean the string of results: i.e. 001 would represent 2 tails followed by a heads. We can picture this using the tree below, which depicts the possible outcomes when $k = 3$:




Here $S = \{0,1\}$. This is a case of sampling with replacement; multiple coin flips could result in tails (we could pick the element 0 from set $S$ in multiple trials). Order also matters - strings 001 and 010 are considered different outcomes. By the reasoning above (using the first rule of counting) we have a total of $n^k = 2^k$ distinct outcomes, since $n = 2$.

## Rolling Dice

Let's say we roll two dice, so $k = 2$ and $S = \{1,2,3,4,5,6\}$. How many possible outcomes are there? In this setting, ordering matters; obtaining 1 with the first die and 2 with the second is different from obtaining 2 with the first and 1 with the second. We are sampling with replacement, since we can obtain the same result on both dice.

The setting is therefore the same as the coin flipping example above (order matters and we are sampling with replacement), so we can use the first rule of counting in the same manner. The number of distinct outcomes is therefore $n^2 = 6^2 = 36$.

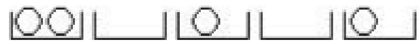
## Sampling with replacement, but where order does not matter

Say you have unlimited quantities of apples, bananas and oranges. You want to select 5 pieces of fruit to make a fruit salad. How many ways are there to do this? In this example, $S = \{1,2,3\}$, where 1 represents apples, 2 represents bananas, and 3 represents oranges. $k = 5$ since we wish to select 5 pieces of fruit. Ordering does not matter; selecting an apple followed by a banana will lead to the same salad as a banana followed by an apple.

This scenario is much more tricky to analyze. It is natural to apply the second rule of counting because order does not matter. So we first pretend that order matters, and then the number of ordered objects is $3^5$ as discussed above. How many ordered options are there for every unordered option? The problem is that this number differs depending on which unordered object we are considering. Let's say the unordered object is an outcome with 5 bananas. There is only one such ordered outcome. But if we are considering 4 bananas and 1 apple, there are 5 such ordered outcomes (represented as 12222,21222,22122,22212,22221).

Now that we see the second rule of counting will not help, can we look at this problem in a different way? Let us first generalize back to our original setting: we have a set $S = \{1,2,\cdots,n\}$ and we would like to know how many ways there are to choose multisets (sets with repetition) of size $k$. Remarkably, we can model this problem in terms of binary strings.

Assume we have 1 bin for each element from set $S$, so $n$ bins. For example, if we selected 2 apples and 1 banana, bin 1 would have 2 elements and bin 2 would have 1 element. In order to count the number of multisets, we need to count how many different ways there are to fill these bins with $k$ elements. We don't care about the order of the bins themselves, just how many of the $k$ elements each bin contains. Let's represent each of the $k$ elements by a 0 in the binary string, and separations between bins by a 1. Consider the following picture:



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This would be a sample placement where $S = \{1, \cdots, 5\}$ and $k = 4$. Counting the number of multi sets is now equivalent to counting the number of placements of the $k$ 0's. We have just reduced what seemed like a very complex problem to a question about a binary string, simply by looking at it from a different perspective!

How many ways can we choose these locations? The length of our binary string is $k+n-1$, and we are choosing which $k$ locations should contain 0's. The remaining $n-1$ locations will contain 1's. Once we pick a location for one zero, we cannot pick it again; repetition is not allowed. Picking location 1 followed by location 2 is the same as picking location 2 followed by location 1, so ordering does not matter. It follows that all we wish to compute is the number of ways of picking $k$ elements from $k+n-1$ elements, without replacement and where the order of placement does not matter. This is given by $\binom{n+k-1}{k}$, as discussed in the Counting Sets section above. This is therefore the number of ways in which we can choose multisets of size $k$ from set $S$.

Returning to our example above, the number of ways of picking 5 pieces of fruit is exactly $\binom{3+5-1}{5} = \binom{7}{5}$

Notice that we started with a problem which seemed very different from previous examples, but, by viewing it from a certain perspective, we were able to use previous techniques (those used in counting sets) to find a solution! This is key to many combinatorial arguments as we will explore further in the next section.

# Combinatorial Proofs

Combinatorial arguments are interesting because they rely on intuitive counting arguments rather than algebraic manipulation. We can prove complex facts, such as $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \cdots + \binom{k}{k}$. You can directly verify this identity by algebraic manipulation. But you can also do this by interpreting what each side means as a combinatorial process. The left hand side is just the number of ways of choosing a $k+1$ element subset from a set of $n$ items. Let us think about a different process that results in the choice of a $k+1$ element subset. We start by picking the lowest numbered element in the subset. It is either the first element of the set or the second or the third or ... If we choose the first element, we must now choose $k$ elements out of the remaining $n-1$ which we can do in $\binom{n-1}{k}$ ways. If instead the lowest numbered element we picked is the second element then we still have to choose $k$ elements from the remaining $n-1$ which can be done in $\binom{n-2}{k}$ ways. Moreover all these subsets are distinct from those where the lowest numbered element was the first one. So we should add the number of ways of choosing each to the grand total. Proceeding in this way, we split up the process into cases according to the first (i.e., lowest-numbered) object we select, to obtain:

$$\textbf{First element selected is either} \begin{cases} \text{element } 1, & \binom{n-1}{k} \\ \text{element } 2, & \binom{n-2}{k} \\ \text{element } 3, & \binom{n-3}{k} \\ \quad \vdots & \\ \text{element}(n-k), & \binom{k}{k} \end{cases}$$

(Note that the lowest-numbered object we select cannot be higher than $n-k$ as we have to select $k$ distinct objects.)

The last combinatorial proof we will do is the following: $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n$. To see this, imagine that we have a set $S$ with $n$ elements. On the left hand side, the $i^{th}$ term counts the number of ways of choosing a subset of $S$ of size exactly $i$; so the sum on the left hand side counts the total number of subsets (of any size) of $S$.

We claim that the right hand side ($2^n$) does indeed also count the total number of subsets. To see this, just identify a subset with an $n$-bit vector, where in each position $j$ we put a 1 if the $j$th element is in the subset, and a 0 otherwise. So the number of subsets is equal to the number of $n$-bit vectors, which is $2^n$ (there are 2 options for each bit). Let us look at an example, where $S=\{1,2,3\}$ (so $n=3$). Enumerate all $2^3=8$ possible subsets of $S$: $\{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. The term $\binom{3}{0}$ counts the number of ways to choose a subset of $S$ with 0 elements; there is only one such subset, namely the empty set. There are $\binom{3}{1}=3$ ways of choosing a subset with 1 element, $\binom{3}{2}=3$ ways of choosing a subset with 2 elements, and $\binom{3}{3}=1$ way of choosing a subset with 3 elements (namely, the subset consisting of the whole of $S$). Summing, we get $1+3+3+1=8$, as expected.