1. **Variations**

   Suppose we were trying to prove $P(n)$ is true for all $n \in N$ by induction on n. Instead we succeeded in proving $\forall k \in N$ if $P(k)$ is true then $P(k+2)$ is true. For each of the following assertions below, state whether (A) it must always hold, or (N) it can never hold, or (C) it can hold but need not always. Give a very brief (one or two sentence) justification for your answers. The domain of all quantifiers is the natural numbers.

   (a) $\forall n \geq 0\ P(n)$.

   (b) If $P(0)$ is true then $\forall n\ P(n+2)$ is true.

   (c) If $P(0)$ is true then $\forall n\ P(2n)$ is true.

   (d) $\forall n\ P(n)$ is false.

   (e) If $P(0)$ and $P(1)$ are true then $\forall n\ P(n)$ is true.

   (f) We can conclude that ($\forall n \leq 10\ \ P(n)$ is true), and ($\forall n > 10\ P(n)$ is false).

2. **Chocolate!**

   Chocolate often comes in rectangular bars marked off into smaller squares. It is easy to break a larger rectangle into two smaller rectangles along any of the horizontal or vertical lines between the squares. Suppose I have a bar containing k squares and wish to break it down into its individual squares. Prove that *no matter which way I break it*, it will take exactly $k-1$ breaks to do this.

3. **Recursion**

   Let the function g be defined recursively on the natural numbers as follows:
   $g(0) = 0, g(1) = 1$, and $g(n) = 5g(n-1) - 6g(n-2)$, for all $n \geq 2$. Show that $\forall n \in N, g(n) = 3^n - 2^n$.

4. **Some Identities by Induction.**

   (a) For $n \in N$ with $n \geq 2$, define s_n by

   $$s_n = (1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{n}).$$

   Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

   (b) Let $a_n = 3^{n+2} + 4^{2n+1}$. Prove that 13 divides a_n for every $n \in N$. (Hint: What can you say about $a_{n+1} - 3a_n$?)

   (c) Prove that $2^n < n!$ for all integers $n \geq 4$.

   (d) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for all integers $n \in \mathbb{N}$.

5. **A pizza proof.**

   Working at the local pizza parlor, I have a stack of unbaked pizza doughs. For a most pleasing presentation, I wish to arrange them in order of size, with the largest pizza on the bottom. I know how to place my spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). This is the only move I know that can change the order of the stack; however, I am willing to keep repeating this move until I get the stack in order. Is it always possible to get the pizzas in order? Prove your answer.

6. **Grading proofs**

   Assign a grade of A (correct) or F (failure) to each of the following proofs. If you give a F, please explain exactly everything that is wrong with the structure or the reasoning in the *proof*. You should justify all your answers (remember, saying that the claim is false is *not* a justification).

   (a) **Claim**: For every $n \in \mathbf{N}$ with $n \geq 1$, n^2+n is odd.
   Proof:
   The proof will be by induction.
   Base case: The natural number 1 is odd.
   Inductive step: Suppose $k \in \mathbf{N}$ and k^2+k is odd. Then,

   $$(k+1)^2+(k+1) = k^2+2k+1+k+1 = (k^2+k)+(2k+2)$$

   is the sum of an odd and an even integer. Therefore, $(k+1)^2+(k+1)$ is odd. By the Principle of Mathematical Induction, the property that n^2+n is odd is true for all natural numbers n.

   (b) **Claim**: For all $x,y,n \in \mathbf{N}$, if $\max(x,y)=n$, then $x=y$.
   Proof:
   The proof will be by induction.
   Base case: Suppose that $n=0$. If $\max(x,y)=0$ and $x,y \in N$, then $x=0$ and $y=0$, hence $x=y$.
   Induction step: Assume that, whenever we have $\max(x,y)=k$, then $x=y$ must follow. Next suppose x,y are such that $\max(x,y)=k+1$. Then it follows that $\max(x-1,y-1)=k$, so by the inductive hypothesis, $x-1=y-1$. In this case, we have $x=y$, completing the induction step.

   (c) **Claim**: $\forall n \in \mathbf{N}.\ n^2 \leq n$.
   Proof:
   The proof will be by induction.
   Base case: When $n=0$, the statement is $0^2 \leq 0$ which is true.
   Induction step: Now suppose that $k \in \mathbf{N}$, and $k^2 \leq k$. We need to show that

   $$(k+1)^2 \leq k+1$$

   Working backwards we see that:

   $$\begin{aligned}(k+1)^2 &\leq k+1\\ k^2+2k+1 &\leq k+1\\ k^2+2k &\leq k\\ k^2 &\leq k\end{aligned}$$

   So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in N$ we know that $n^2 \leq n$.