CS 70 Discrete Mathematics and Probability Theory
Fall 2013 Vazirani Conditional Probability Practice

1. **Independence**

   We flip two unbiased coins: a nickel and a dime, and consider the following events:

   (a) The nickel comes up heads.
   (b) The dime comes up heads.
   (c) The nickel and dime both come up heads.
   (d) Exactly one of the nickel and dime comes up heads.
   (e) The nickel and dime both come up the same way.

   State without proof whether each of the following pairs of events are independent:

   - (a) and (b):
   - (a) and (c):
   - (a) and (d):
   - (a) and (e):
   - (c) and (b):
   - (d) and (b):
   - (e) and (b):
   - (c) and (d):
   - (c) and (e):
   - (d) and (e):

   State without proof whether each of the following triples of events are independent:

   - (a), (b), (c):
   - (a), (b), (d):
   - (a), (b), (e):
   - (b), (d), (e):
   - (a), (c), (e):

2. **Correlation**

   It was suggested in class that, when $\Pr[A|B] > \Pr[A]$, then A and B may be viewed intuitively as being positively correlated. One might wonder whether "being positively correlated" is a symmetric relation. Prove or disprove: If $\Pr[A|B] > \Pr[A]$ holds, then $\Pr[B|A] > \Pr[B]$ must necessarily hold, too. (You may assume that both $\Pr[A|B]$ and $\Pr[B|A]$ are well-defined, i.e., neither $\Pr[A]$ nor $\Pr[B]$ are zero.)

3. **Monty Hall Again**

   In the three-door Monty Hall problem, there are two stages to the decision, the initial pick followed by the decision to stick with it or switch to the only other remaining alternative after the host has shown an incorrect door. An extension of the basic problem to multiple stages goes as follow.

   Suppose there are four doors, one of which is a winner. The host says: "You point to one of the doors, and then I will open one of the other non-winners. Then you decide whether to stick with your original pick or switch to one of the remaining doors. Then I will open another (other than the current pick) non-winner. You will then make your final decision by sticking with the door picked on the previous decision or by switching to the only other remaining door.

   (a) How many possible strategies are there?

   (b) For each of the possible strategies, calculate the probability of winning. What is the best strategy?

4. **Smokers**

   A health study tracked a group of people for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers.

   Suppose we select, uniformly at random, a participant from this study, and it turns out that this participant died at some point during the five-year period. Calculate the probability that this participant was classified as a heavy smoker at the beginning of the study. Show your calculation clearly.

5. **The myth of fingerprints**

   A crime has been committed. The police discover that the criminal has left DNA behind, and they compare the DNA fingerprint against a police database containing DNA fingerprints for 20 million people. Assume that the probability that two DNA fingerprints (falsely) match by chance is 1 in 10 million. Assume that, if the crime was committed by someone whose DNA fingerprint is on file in the police database, then it's certain that this will turn up as a match when the police compare the crime-scene evidence to their database; the only question is whether there will be any false matches.

   Let D denote the event that the criminal's DNA is in the database; $\neg D$ denotes the event that the criminal's DNA is not in the database. Assume that it is well-documented that half of all such crimes are committed by criminals in the database, i.e., assume that $\Pr[D] = \Pr[\neg D] = 1/2$. Let the random variable X denote the number of matches that are found when the police run the crime-scene sample against the DNA database.

   (a) Calculate $\Pr[X = 1|D]$.

   (b) Calculate $\Pr[X = 1|\neg D]$.

   (c) Calculate $\Pr[\neg D|X = 1]$. Evaluate the expression you get and compute this probability to at least two digits of precision.

   As it happens, the police find exactly one match, and promptly prosecute the corresponding individual. You are appointed a member of the jury, and the DNA match is the only evidence that the police present. During the trial, an expert witness testifies that the probability that two DNA fingerprints (falsely) match by chance is 1 in 10 million. In his summary statement, the prosecutor tells the jury that this means that the probability that the defendant is innocent is 1 in 10 million.

(d) What is wrong with the prosecutor's reasoning in the summary statement?

(e) Do you think the defendant should be convicted? Why or why not?

6. **Poisoned pancakes**

   You have been hired as an actuary by IHOP corporate headquarters, and have been handed a report from Corporate Intelligence that indicates that a covert team of ninjas hired by Denny's will sneak into some IHOP, and will have time to poison five of the pancakes being prepared (they can't stay any longer to avoid being discovered by Pancake Security). Given that an IHOP kitchen has 50 pancakes being prepared, and there are ten patrons, each ordering five pancakes (which are chosen uniformly at random from the pancakes in the kitchen), calculate the probabilities that the first patron:

   (a) will not receive any poisoned pancakes;

   (b) will receive exactly one poisoned pancake;

   (c) will receive at least one poisoned pancake;

   (d) will receive at least one poisoned pancake given that the second patron received at least one poisoned pancake;

   (e) Calculate the probability that any of the first three receive at least one poisoned pancake.

7. **Colorful coins**

   We are given three coins. The first coin is a fair coin painted blue on the heads side and white on the tails side. The other two coins are biased so that the probability of heads is p. They are painted blue on the tails side and red on the heads side. One coin is randomly chosen and flipped twice.

   (a) Describe the outcomes in the sample space, and give their probabilities. [NOTE: You may want to draw a tree to illustrate the sample space.]

   (b) Now suppose two coins are chosen randomly *with replacement* and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in part (a)? [NOTE: You may want to draw a tree to illustrate the sample space.]

   (c) Now suppose two coins are chosen randomly *without replacement* and each flipped once. Describe the outcomes in the sample space in this new experiment, and give their probabilities. Are they the same as in parts (a) or (b)? [NOTE: You may want to draw a tree to illustrate the sample space.]

   (d) Suppose the probability that the two sides that land face up are the same color is $\frac{29}{96}$ in the experiment in part (c). What does this tell you about the possible values of p?

   (e) Let A be the event that you get a head on the first flip and B is the event that you get a head on the second flip. In each of the experiments in (a), (b) and (c), determine whether A and B are independent events.

8. **A paradox in conditional probability?**

   Here is some on-time arrival data for two airlines, A and B, into the airports of Los Angeles and Chicago. (Predictably, both airlines perform better in LA, which is subject to less flight congestion and less bad weather.)

	Airline A		Airline B	
	# flights	# on time	#flights	# on time
Los Angeles	600	534	200	188
Chicago	250	176	900	685

(a) Which of the two airlines has a better chance of arriving on time into Los Angeles? What about Chicago?

(b) Which of the two airlines has a better chance of arriving on time overall?

(c) Do the results of parts (a) and (b) surprise you? Explain the apparent paradox, and interpret it in terms of conditional probabilities.

9. **A flippant choice**

We have noted that if a fair coin is flipped three times, there are eight equally probable outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Two CS 70 students play a game based on coin flipping. Player A selects one of the triplets just listed; player B selects a different one. The coin is then repeatedly flipped until one of the chosen triplets appears as a run and wins the game. For example, if player A chooses HHT and player B chooses THT and the flips are THHHT, player A wins.

Fill in the table below to show player B's best choice of triplet for each possible choice that player A makes, and the probability of player B winning with a best choice. Then explain why the odds for one player winning are so lopsided.

Player A's choice	Player B's best choice	Player B's probability of winning
HHH		
HHT		
HTH		
HTT		
THH		
THT		
TTH		
TTT		

10. **Stakes well done**

Two players, Alice and Bob, each stake 32 pistoles on a three-point, winner-take-all game of chance. The game is played in rounds; at each round, one of the two players gains a point and the other gains none. Normally the first player to reach 3 points would win the 64 pistoles. However, it starts to rain during the game, and play is suspended at a point where Alice has 2 points and Bob has 1 point. Alice and Bob have to figure out how to split the money.

You should assume that Alice and Bob are evenly matched, so that in each round Alice and Bob each have a 50% chance of winning the round. Assume also that Alice's share should be proportional to the conditional expected valueof her winnings (specifically, her winnings if the game were continued to the end from this point). The same goes for Bob.

Calculate a fair way to distribute the 64 pistoles using this notion of fairness. How many pistoles does Alice receive? Bob?