1. **Square**

   Given a point chosen uniformly at random over a unit square (of sidelength 1), what is the density function of the random variable, X, corresponding to the distance to its border?

   Find the expected value of X.

2. **Random arrivals**

   Consider two people arriving uniformly at random in a time interval of one hour. We wish to bound how long the first has to wait for the second.

   Consider choosing two points X and Y independently and uniformly from the unit interval.

   (a) What is the joint density function $f(x,y)$? (Hint: it's not complicated.)

   (b) What is the expected value of $|x-y|$? (This is the expected time between arrivals.)

3. **Linearity of Expectation Again**

   Linearity of expectation holds for discrete r.v. Show that linearity of expectation holds for continuous random variables as well.

4. **Lunch Date**

   Alice and Bob agree to try to meet for lunch between 12 and 1pm at their favorite sushi restaurant. Being extremely busy they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving. What is the probability that they will actually meet for lunch? Phrase your solution using the language of continuous random variables.

5. **Cumulative Distribution Function**

   In class, the statistics of a random variable are specified by the distribution in the discrete case and specified by the probability density function (pdf) in the continuous case. To unify the two cases, we can define the *cumulative distribution function*(cdf) F for a r.v., which is valid for both discrete and continuous r.v.'s:

$$F(a) := \Pr[X \leq a], \qquad a \in \Re.$$

   (a) In the discrete case, show that the cdf of a r.v. contains exactly the same information as its distribution, by expressing F in terms of the distribution and expressing the distribution in terms of F.

   (b) In the continuous case, show that the cdf of a r.v. contains exactly the same information as its pdf, by expressing F in terms of the pdf and expressing the pdf in terms of F.

(c) Compute and plot the cdf for (i) $X \sim \text{Geom}(p)$, (ii) $X \sim \text{Exp}(\lambda)$.

(d) Identify two key properties that a cdf of any r.v. has to satisfy.

6. **Memorylessness**

We begin by proving two very useful properties of the exponential distribution. We then use them to solve a problem about the expected life of a package of batteries.

(a) Let r.v. X have geometric distribution with parameter p. Show that, for any positive m,n, we have

$$\Pr[X > m+n \mid X > m] = \Pr[X > n].$$

This is the "memoryless" property of the geometric distribution. Why do you think this property is called memoryless?

(b) Let r.v. X have exponential distribution with parameter λ. Show that, for any positive s,t, we have

$$\Pr[X > s+t \mid X > t] = \Pr[X > s].$$

[This is the "memoryless" property of the exponential distribution.]

(c) Let r.v.'s X_1, X_2 be independent and exponentially distributed with parameters λ_1, λ_2. Show that the r.v. $Y = \min\{X_1, X_2\}$ is exponentially distributed with parameter $\lambda_1 + \lambda_2$. [Hint: work with cdf's.]

(d) You have a digital camera that requires two batteries to operate. You purchase n batteries, labeled $1, 2, \ldots, n$, each of which has a lifetime that is exponentially distributed with parameter λ and is independent of all the other batteries. Initially you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer.

(e) In the scenario of part (c), what is the probability that battery i is the last remaining working battery, as a function of i?

7. **A difference between discrete and continuous r.v.'s**

Discrete and continuous r.v.'s have a lot of similarities but some differences too.

(a) Suppose X is a discrete r.v. Let the r.v. $Y = cX$ for some constant c. Express the distribution of Y in terms of that of X.

(b) Suppose X is a continuous r.v. Let the r.v. $Y = cX$ for some constant c. Express the pdf of Y in terms of that of X. Is there any difference with the discrete case? [Hint: work with cdf's introduced in Question 5 on this handout.]

(c) If $X \sim N(\mu, \sigma^2)$, what is the density of $Y = cX$?

8. **Normal Distribution**

If a set of grades on a Discrete Math examination in an inferior school (not UC!) are approximately normally distributed with a mean of 64 and a standard deviation of 7.1, find:

(a) the lowest passing grade if the bottom 5% of the students fail the class;

(b) the highest B if the top 10% of the students are given A's.

NOTE: You may assume that if X is normal with mean 0 and variance 1, then $\Pr[X \leq 1.3] \approx 0.9$ and $\Pr[X \leq 1.65] \approx 0.95$.