CS 70 Discrete Mathematics and Probability Theory
Fall 2013 Vazirani Counting + Sample Spaces Practice

1. **Counting, counting and counting**

   The only way to learn counting is to practice, practice, practice, so here is your chance to do so. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

   (a) How many 10-bit strings are there that contain exactly 4 ones?

   (b) How many different 13-card bridge hands are there? (A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.)

   (c) How many different 13-card bridge hands are there that contain no aces?

   (d) How many different 13-card bridge hands are there that contain all four aces?

   (e) How many different 13-card bridge hands are there that contain exactly 6 spades?

   (f) How many 99-bit strings are there that contain more ones than zeros?

   (g) If we have a standard 52-card deck, how many ways are there to order these 52 cards?

   (h) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

   (i) How many different anagrams of FLORIDA are there? (An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.)

   (j) How many different anagrams of ALASKA are there?

   (k) How many different anagrams of ALABAMA are there?

   (l) How many anagrams does the word PAPASAN have where the S and N are not adjacent? For example, count APPSANA but do not count APAANSP.

   (m) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins?

   (n) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty?

   (o) How many different ways are there to throw 9 identical balls into 27 bins?

   (p) How many ways are there to place 50 unlabeled balls in 9 labeled bins where each bin contains at least as many balls as its bin number. (That is, bin 1 contains at least 1 ball, bin 2 contains at least 2, and so on.)

   (q) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

2. **More Counting**

   Let A, B be finite sets, with $|A| = m$ and $|B| = n$.

(a) How many subsets does A have? (Recall that the empty set and A are both subsets of A.)

(b) How many distinct functions $f : A \to B$ are there from A to B?

(c) Suppose $m = n$. How many distinct bijections are there from A to B?

3. **And More Counting**

How many non-negative integer solutions $(x_1, \ldots, x_7)$ are there to the following equation?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 2003$$
$$x_1 \geq 0, \ldots, x_7 \geq 0, \qquad x_1, \ldots, x_7 \in \mathbb{Z}$$

Order matters. For instance, $(1, 2002, 0, 0, 0, 0, 0)$ counts as a different solution than $(2002, 1, 0, 0, 0, 0, 0)$.

4. **Sum of digits**

Choose a number uniformly at random between 0 and 999,999, inclusive. What is the probability that the digits sum to 19?

5. **Algebraic vs. combinatorial proofs**

Consider the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(a) Prove the identity by algebraic manipulation (using the formula for the binomial coefficients).

(b) Prove the identity using a combinatorial argument. (Write both sides as the answer to a question of the form "how many ways can you...?")

6. **Red cards**

Consider a deck with just the four aces (red: hearts, diamonds; black: spades, clubs). Melissa shuffles the deck and draws the top two cards.

Given that Melissa has the ace of hearts, what is the probability that Melissa has both red cards?

Given that Melissa has at least one red card, what is the probability that she has both red cards?

7. **Sample Space and Events**

Consider the sample space Ω of all outcomes from flipping a coin 4 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a Heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a Heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first flip and the third flip are both Heads. List all the outcomes in C. How many are there?

(e) Let D be the event that the first flip or the third flip is a Heads. List all the outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express the event C in terms of A and B. Express the event D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 4 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

8. **Probability Models**

Suppose you have two coins, one is biased with a probability of p coming up Heads, and one is biased with a probability of q coming up Heads. Answer the questions below, but you don't need to provide justifications.

(a) Suppose $p = 1$ and $q = 0$.

i. You pick one of the two coins randomly and flip it. You repeat this process n times, each time randomly picking one of the two coins and then flipping it. Consider the sample space Ω of all possible length n sequences of Heads and Tails so generated. Give a reasonable probability assignment (i.e. assign probabilities to all the outcomes) to model the situation.

ii. Now you pick one of the two coins randomly, but flip the *same* coin n times. Identify the sample space for this experiment together with a reasonable probability assignment to model the situation. Is your answer the same as in the previous part?

(b) Repeat the above two questions for arbitrary values of p and q. Express your answers in terms of p and q.