CS 70	Discrete Mathematics and Probability Theory
Fall 2013	Vazirani — Stable Marriage Practice

1. **Stable Marriage True-or-False?**

   For each of the following claims, state whether the claim is true or false. If it is true, give a *short* proof; if it is false, give a *simple* counterexample.

   (a) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.

   (b) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.

   (c) For every $n > 1$, there is a stable marriage instance with n men and n women which has an unstable pairing in which every unmatched man-woman pair is a rogue couple.

   (d) An instance of the stable marriage problem (i.e., a set of men and women and their associated preference lists) must have at least two different stable pairings, a male-optimal one and a female-optimal one.

   (e) In any instance of the stable marriage problem, if woman w is matched to her first choice by the traditional marriage algorithm, then she must be matched to her first choice in all stable pairings.

   (f) If man m is rejected by woman w at some point in the traditional marriage algorithm, then no stable pairing exists in which m is matched to w.

2. **Stable marriage**

   Consider a set of four boys (a, b, c, d) and four girls (1, 2, 3, 4) with the preferences shown below.

boy	preferences	girl	preferences
a	1>2>3>4	1	d>b>c>a
b	2>1>4>3	2	a>d>b>c
c	1>3>2>4	3	a>b>c>d
d	2>1>3>4	4	d>c>a>b

   (a) Run the traditional marriage algorithm on this instance. Show each stage of the algorithm, and give the resulting matching, expressed as a set of boy-girl pairs. You can do this by hand, or you can try to program it!

   (b) The matching you found above is boy-optimal. Find now a girl-optimal stable matching. Compare the two matchings.

3. **Cold feet and Eager Beavers.**

   Consider a slightly different setting for the stable marriage algorithm: suppose that some of the men are nervous about proposing, and it takes them extra time to work up the courage to ask the women to marry them. Other men are overly eager, and do not have the courtesy to wait 24 hours between when they were rejected and when they next propose. The result of this is that some men might procrastinate for several days, while others might propose and get rejected several times in a single day. Can the order of the proposals change the resulting pairing? Give an example of such a change or prove that the pairing that results is the same.

4. **Man-Optimal, Woman-Optimal**

   In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable pairings, $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$. Also, each woman W has a different partner in the three pairings. Therefore, each woman has a clear preference ordering of the three pairings (according to the ranking of her partners in her preference list). Now, suppose that for woman W_1 this order is $\mathscr{P}_1 > \mathscr{P}_2 > \mathscr{P}_3$. True or false: every woman has the same preference ordering $\mathscr{P}_1 > \mathscr{P}_2 > \mathscr{P}_3$. Justify your answer carefully, using facts about Stable Marriage that we proved in class.

5. **One side wins, the other loses**

   Mr. and Mrs. Matchmaker are at work in their matchmaking agency. There are n men and n women, each having strict preferences over people of the opposite gender as suitors for marriage. The Matchmakers want to produce a stable pairing.

   Minions who work for the matchmakers have produced two proposals each detailing one set of pairings. Mr. Matchmaker proposes the following scheme to produce the final pairings which are announced to the clients: for each man look at all of the women who are matched with him in at least one of the proposals (there could be one or two women). Then match this man with the best (according to his preferences) of these women.

   (a) Prove that Mr. Matchmaker's scheme actually results in a matching in which no two men are matched with the same woman.

   (b) Prove that the matching produced is stable.

   (c) Mrs. Matchmaker is very suspicious of this scheme and she thinks that it will do a terrible job for women. Help her by proving that Mr. Matchmaker's scheme results in a matching in which each woman is matched to her least favorite suitor among the two proposals produced by the minions.

   (d) In class you learned that the propose and reject algorithm results in a pairing which is optimal for men and pessimal for women. Give another proof using the things you learned in previous parts that such a pairing which is optimal for men and pessimal for women always exists.

6. **I'm too good to marry**

   In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing would ultimately have to be partial, i.e. some individuals would remain single.

   The notion of stability here should be adjusted a little bit: a pairing is stable if there is no rogue couple, and there is no paired individual who prefers being single over being with his/her pair.

   (a) Prove that the stable marriage algorithm still yields a stable pairing. You can approach this by introducing imaginary mates (one for each person) that people marry if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?

(b) As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact using Mr. Matchmaker's scheme and its properties from the previous question (you don't need to have proved them to use the results here).