EE 123 (Fall 2006) Digital Signal Processing **Discussion: 3**

Z-transform and LCCDE

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1 Z-transform and ROC

- Finite length sequence
- Right-handed sequence

2 Linear Constant Coefficient Difference Equations

- Example: $y[n] = ay[n-1] + x[n]$

- Solving LCCDE

- Initial Rest Conditions $\Rightarrow$ LTI + Causal

In general, we have:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

First, let's consider a causal input $x[n]$, such that $x[n] = 0 \;\forall\; n < n_0$. Initial rest conditions imply that we will set the output $y[n] = 0 \;\forall\; n_0 - N \leq n \leq n_0 - 1$. Then, the resulting system is causal, since for $n < n_0$, $y[n] = 0$ and for $n \geq n_0$, $y[n]$ only depends on past values of input, and past values of output, which themselves only depend on past values of input.

Time-invariance follows easily as well. Let's consider $x'[n] = x[n-D]$. Now, from above, we have:

$$\begin{aligned} y[n-D] &= \sum_{k=1}^{N} a_k y[n-D-k] + \sum_{k=0}^{M} b_k x[n-D-k] \\ \Rightarrow y'[n] &= \sum_{k=1}^{N} a_k y'[n-k] + \sum_{k=0}^{M} b_k x'[n-k] \end{aligned}$$

where $y'[n] = y[n-D]$. Therefore, $y'[n]$ is a solution for the original difference equation, and is also the unique solution that satisfies the initial rest conditions.

Assume that $x_1[n]$ and $x_2[n]$ are causal signals, such that $x_i[n] = 0 \;\forall\; n < n_i$. Without loss of generality, assume $n_1 < n_2$. Let $y_i[n]$ be the unique solution for the input $x_i[n]$ satisfying the initial rest conditions. Then, we see that:

$$\begin{aligned} c_1 y_1[n] + c_2 y_2[n] &= c_1 \left(\sum_{k=1}^{N} a_k y_1[n-k] + \sum_{k=0}^{M} b_k x[n-k] \right) + c_2 \left(\sum_{k=1}^{N} a_k y_1[n-k] + \sum_{k=0}^{M} b_k x[n-k] \right) \\ &= \sum_{k=1}^{N} a_k \left(c_1 y_1[n-k] + c_2 y_2[n-k] \right) + \sum_{k=0}^{M} b_k \left(c_1 x_1[n-k] + c_2 x_2[n-k] \right) \end{aligned}$$

Hence, $c_1 y_1[n] + c_2 y_2[n]$ is a solution for the difference equation with $c_1 x_1[n] + c_2 x_2[n]$ as input. Furthermore, it is the unique solution that satisfies the initial rest conditions, since on their own, $y_i[n] = 0 \;\forall\; n < n_i$, and so, under our assumption that $n_1 < n_2$, $c_1 y_1[n] + c_2 y_2[n] = 0 \;\forall\; n < n_1$.

In conclusion, a LCCDE with initial rest conditions gives us a causal and LTI system.