Discrete Fourier Series

$x(t)$ (real.) continuous $\xrightarrow{\text{C.T.F.T.}}$ $X(\Omega)$ (real.) C.T.F.T $= \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$

$x(n)$ (int.) discrete time signal (∞ or finite length) $\xrightarrow{\text{D.T.F.T}}$ $X(\omega) = \sum_n x(n)\, e^{-j\omega n}$ (ω real.)

(∞ly long, finite long.) $x(n)$ (int) discrete time $\xrightarrow{\text{Z.T}}$ $X(z) = \sum_n x(n)\, z^{-n}$ (z complex.)

$\tilde{x}(n)$ (int) periodic. $\xrightarrow{\text{D.F. Series.}}$ $\tilde{X}(k) = \sum_{n \text{ over one period}} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$

$x(n)$ (int) finite length. $\xrightarrow{\text{D.F.T.}}$ $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}}$ (k int, finite length)

DFS = Discrete Fourier Series

Deals with $\tilde{x}(n)$ periodic, discrete Time signal.

$$\tilde{x}(n) = \tilde{x}(n + kN)$$

k any integer = period.

Idea: Decompose $\tilde{x}(n)$ in Terms of exponentials. periodic with period N.

$$e_k(n) = e^{j\frac{2\pi nk}{N}} \qquad k = 0, \ldots, N-1$$

$\mathrm{Re}[e_k(n)]$

There are N, periodic exponentials with period N.

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi nk}{N}}$$

weights

$e_k(n)$ is periodic with period N:

$$e_k(n) \overset{??}{=} e_{k+rN}(n)$$

r: arbitrary int.

<u>proof</u>:

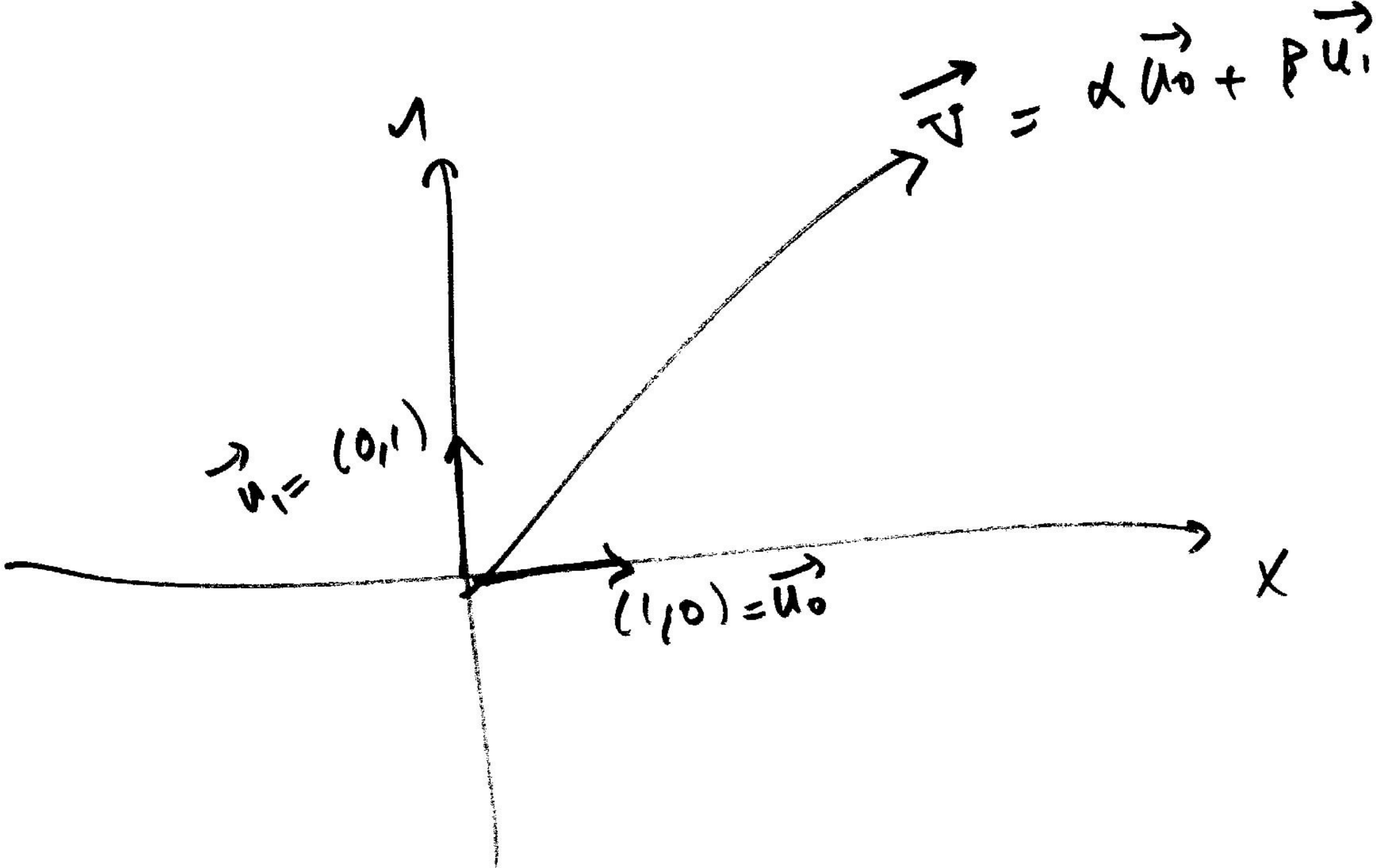
$$e^{j2\pi nk/N} \overset{?}{=} e^{j2\pi n\left(\frac{k+rN}{N}\right)}$$

$$= e^{j2\pi kn/N}\, e^{j2\pi n \frac{rN}{N}}$$

$$= e^{j2\pi kn/N}$$

$$e_0(n) = e_N(n) = e_{2N}(n) = e_{3N}(n) = \dots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \dots$$



$\vec{v} = \alpha \vec{u_0} + \beta \vec{u_1}$
y
$\vec{u_1} = (0,1)$
$(1,0) = \vec{u_0}$
x

$\underline{Q}$ How find "weights"? $\bar{X}(k)$?

proposal:

$$\bar{X}(k) \overset{??}{=} \sum_{n=0}^{N-1} \underbrace{\hat{x}(n)}\, e^{-j\frac{2\pi nk}{N}}$$

proof:

$$\bar{X}(k) \overset{??}{=} \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{l=0}^{N-1} \bar{X}(l)\, e^{j\frac{2\pi n l}{N}} \right) e^{-j\frac{2\pi nk}{N}}$$

$$\overset{??}{=} \sum_{l=0}^{N-1} \bar{X}(l) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)n}{N}} \right]}_{A}$$

$$A = \delta(l-k-rN)$$

what is A?

$$\checkmark \overset{??}{=} \sum_{l=0}^{N-1} \bar{X}(l)\,\delta(l-k-rN) = \bar{X}(k+rN) = \bar{X}(k)$$

r → arb. int.

$\bar{X}(k) = \bar{X}(k+N) = \bar{X}(k+2N)$

Case (1): If $l-k$ is an int. multiple of N.

$l-k = rN$

$$(A) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi rNn}{N}} = 1$$

Case (2) $l-k \neq rN$ ~~is~~ $l-k$ is not an int. multiple of N.

$$(A) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j2\pi\frac{(l-k)n}{N}}$$

Recall $\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$(A) = \frac{1}{N}\,\frac{1-e^{j2\pi\frac{(l-k)N}{N}}}{1-e^{j2\pi\frac{(l-k)}{N}}} = \phi$$

$(A) = \delta(l-k-rN)$

$\bar{X}(K) = X(K + rN)$ $\quad r = $ arb. int.

$\Rightarrow$ $X(K)$ is a periodic sequence with period N.

$\Rightarrow$ From now on refer to $X(K)$ as

$\tilde{X}(K)$

DFS pair.

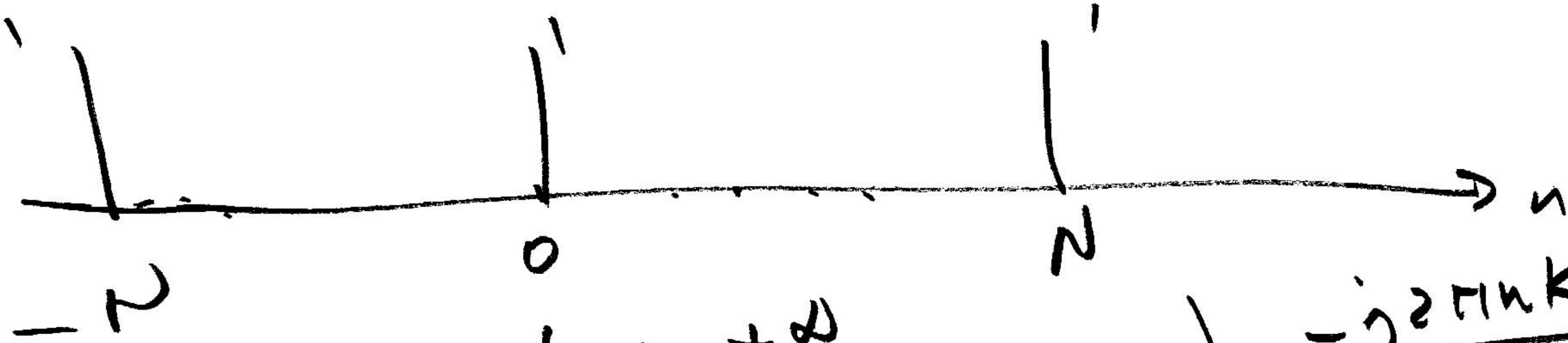
Analysis. $$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$$

synthesis $$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{+j\frac{2\pi nk}{N}}$$

DFS:

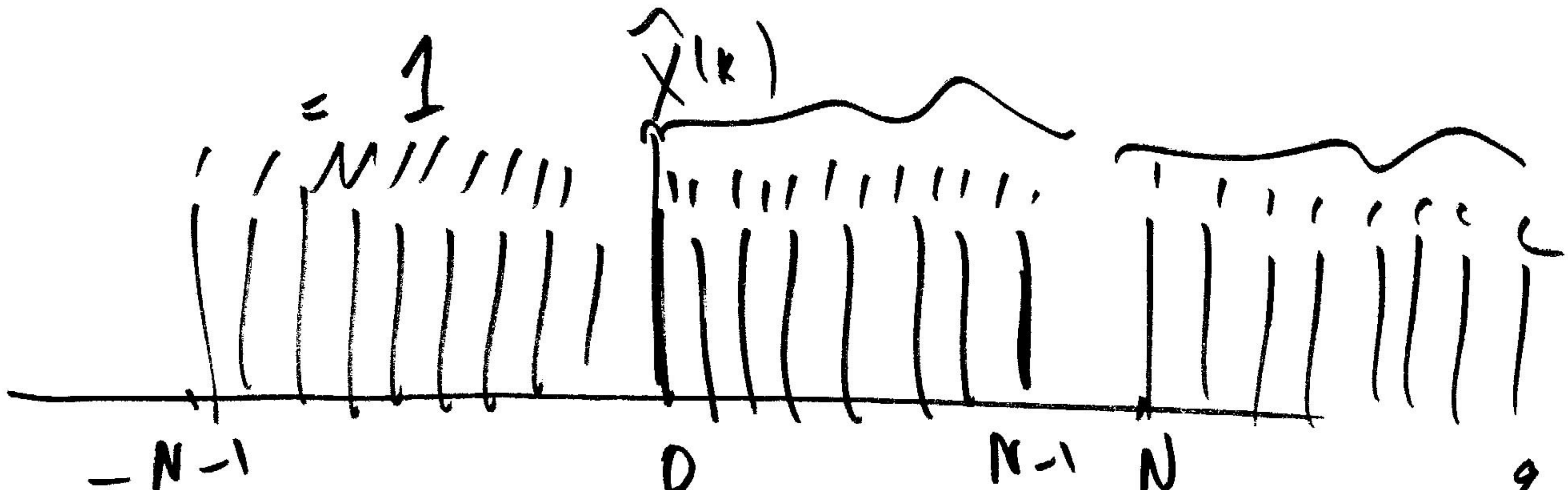
periodic N pt seq $\tilde{x}(n)$ $\longrightarrow$ periodic N point seq in freq. Domain. $\tilde{X}(k)$

Ex $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n + rN)$



$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \delta(n+rN) \right) e^{-j\frac{2\pi nk}{N}}$$

$= 1$



$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}}$$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} \delta(n + rN)$$

equation.
Train.

Shift Property

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$$

$$\tilde{x}(n-n_0) \longleftrightarrow e^{-j\frac{2\pi n_0 k}{N}} \tilde{X}(k)$$

Periodic Convolution:

$$\underset{\text{period } N}{\tilde{x}_1} \underset{\text{period convolution}}{\circledast} \underset{\text{period } N.}{\tilde{x}_2} = \underset{\text{period } N.}{\tilde{x}_3}$$

$$\triangleq \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k)\, \tilde{X}_2(k)$$

<u>DFT</u> = Discrete Fourier Transform.

N pt

$x(n) \longrightarrow X(k)$ — N pt seq

DFS

DFT $\rightarrow$ DFS

DFT $\rightarrow$ DTFT.

First Approach To DFT via DFS

1. Start with a finite extent seq $x(n)$
   N points long $n = 0, \dots, N-1$

2. "Periodicize" $x(n)$ to get $\tilde{x}(n)$ with
   period N.

$$x(n) = \tilde{x}(n) R_N(n)$$

extracts one period of $\tilde{x}(n)$

$$R_N(n) = \begin{cases} 1 & n = 0, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}(n) = \sum_{r=-\infty}^{+\infty} x(n + rN)$$

periodicization

3. Take DFS of $\tilde{x}(n) \rightarrow \tilde{X}(k)$

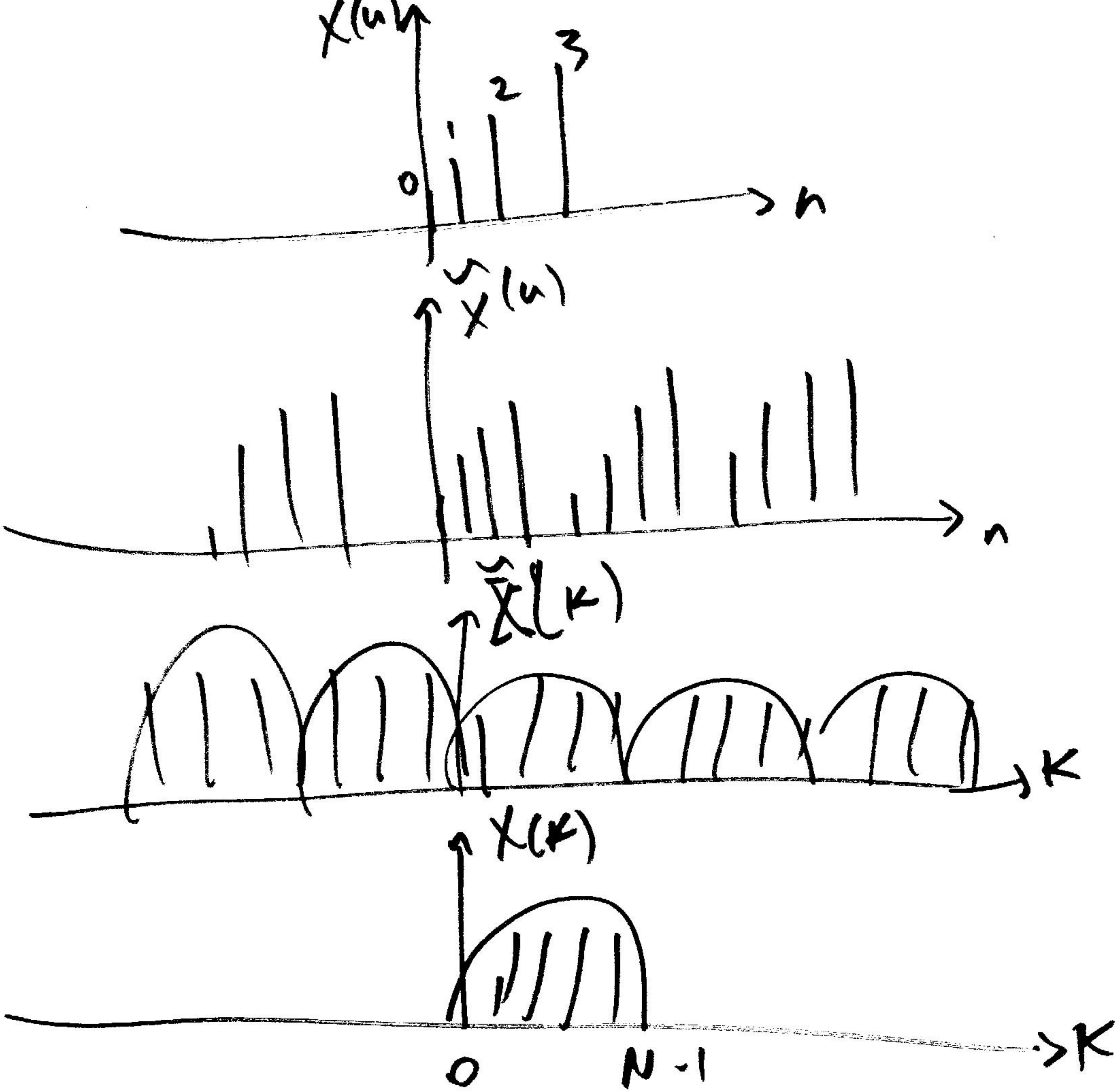
4. Take one period of $\tilde{X}(k)$ to get

$$\bar{X}(k) = \text{DFT of } x(n)$$

$$\bar{X}(k) = \tilde{X}(k) R_N(k)$$

$$\underset{N\,pt}{x(n)} \longrightarrow \underset{\substack{\text{periodic} \\ N\,pt.}}{\tilde{x}(n)} \xrightarrow{\text{DFS}} \underset{\substack{\text{periodic} \\ N\,pt}}{\tilde{X}(k)} \longrightarrow \underset{N\,pt.}{\bar{X}(k)}$$

Ex



Defn of DFT

$$X(k) = N\text{pt DFT of } x(n) = \begin{cases} \displaystyle\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

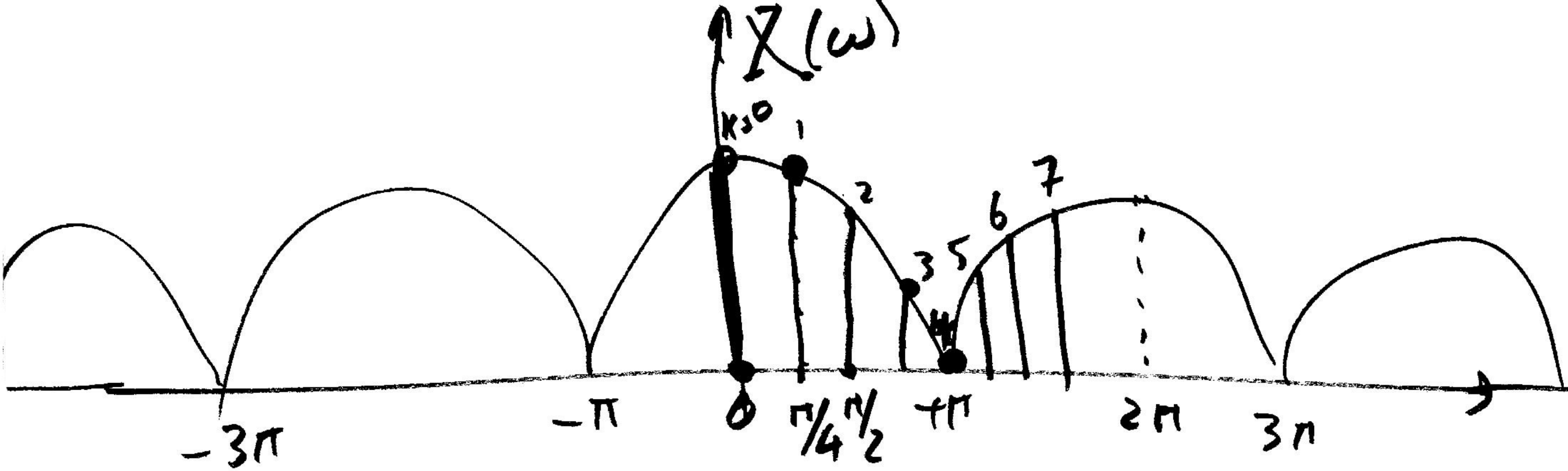
$$x(n) = \begin{cases} \displaystyle\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi n k}{N}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Relate DFT To DTFT:

$$X(\omega) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$$X(k) = \begin{cases} \left[X(\omega)\right]_{\omega = \frac{2\pi k}{N}} & 0 \le k < N \\ 0 & \text{otherwise.} \end{cases}$$

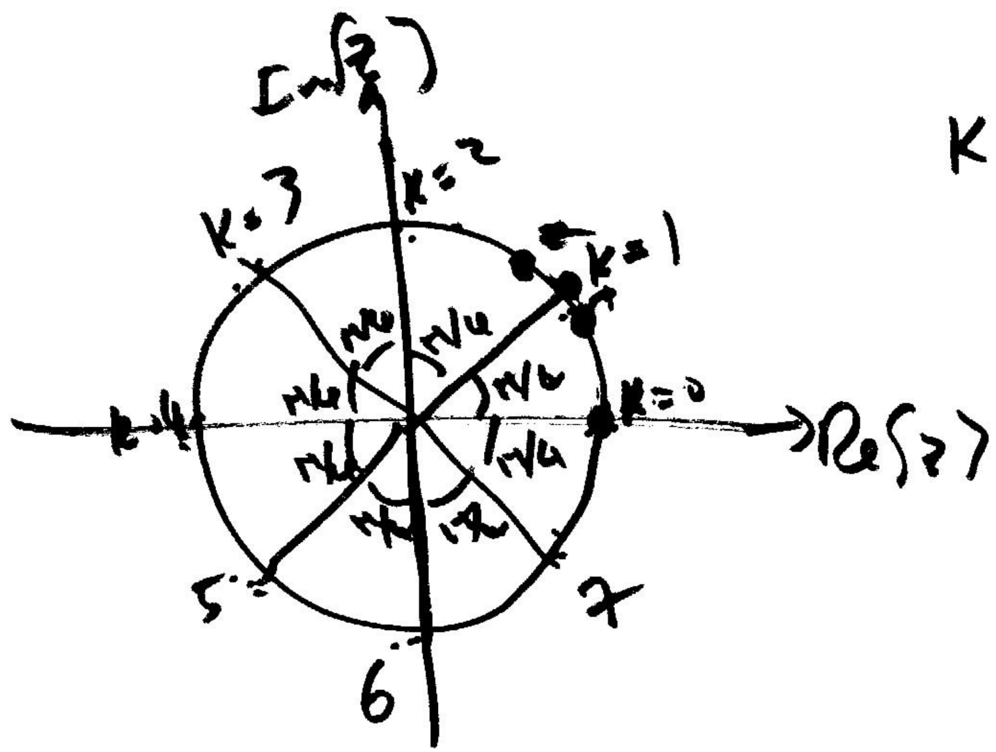
DFT is equally spaced samples of DTFT.



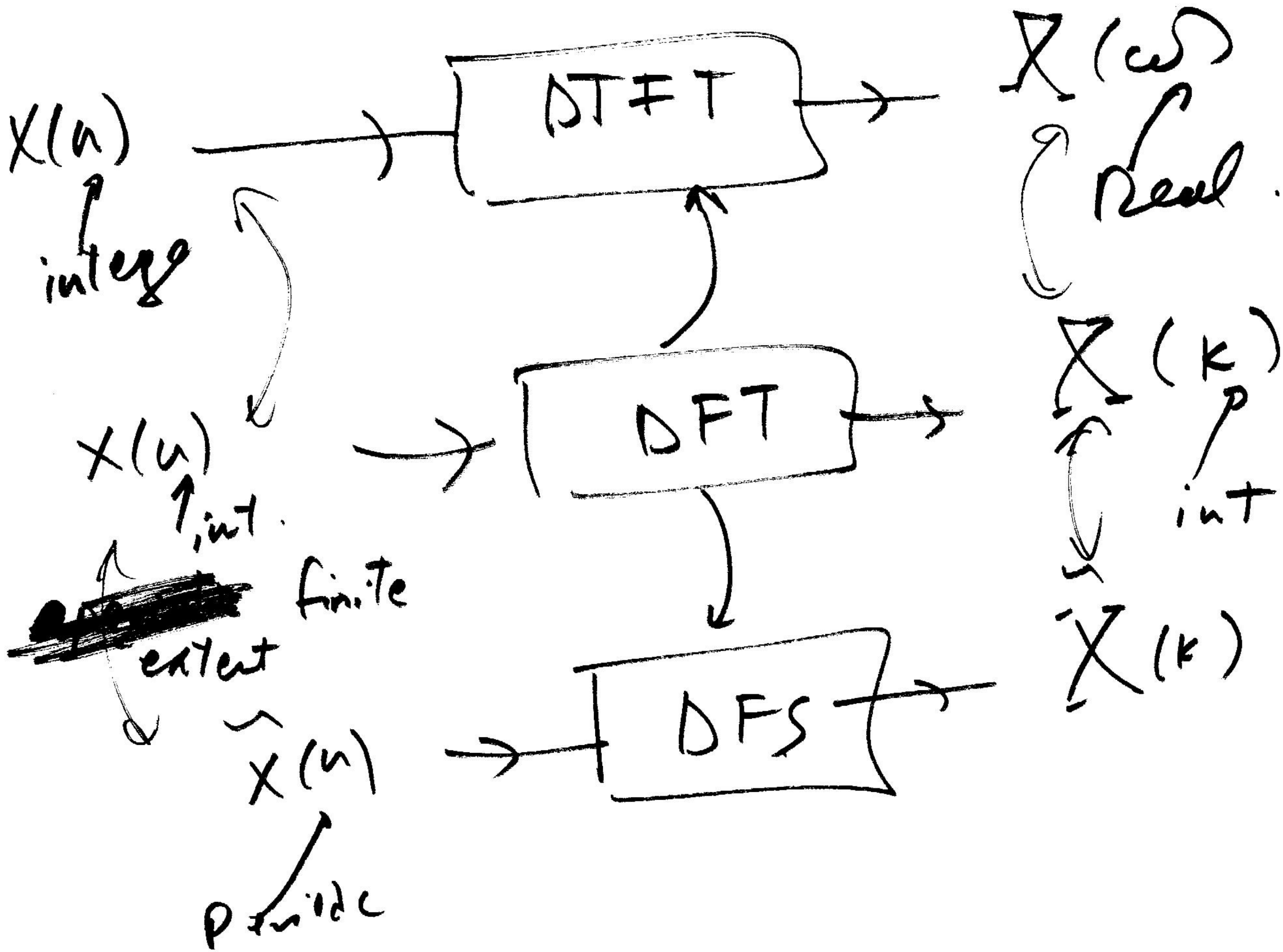
8 pt seq. $\longrightarrow$ 4pt DFT.

$$\omega = \frac{2\pi k}{8}$$

$K = 0, 1, 2, 3, 4, 5, 6, 7$



z plane


x(n)
integer
DTFT
X(ω)
Real
x(n)
int.
finite
extent
DFT
X(k)
int
x̃(n)
periodic
DFS
X̃(k)