Oct. 5, 2006

Discrete Fourier Series

$x(t)$ ← real. continuous $\xrightarrow{\text{C.T.F.T.}}$ $X(\Omega)$ ← real. (C.T.F.T) $= \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$

$x(n)$ ← int. discrete time signal (or finite length) $\xrightarrow{\text{D.T.F.T}}$ $X(\omega) = \sum_n x(n) e^{-j\omega n}$. ($\omega$ ← real.)

$x(n)$ ← int. discrete time (finite length) $\xrightarrow{\text{Z.T}}$ $X(z) = \sum_n x(n) z^{-n}$ (z ← complex.)

$\tilde{x}(n)$ ← int. periodic. $\xrightarrow{\text{D.F. Series.}}$ $\tilde{X}(k) = \sum_{n \text{ over one period}} \tilde{x}(n) e^{-j\frac{2\pi nk}{N}}$

$x(n)$ ← int. finite length. $\xrightarrow{\text{D.F.T.}}$ $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$ (k ← int; finite length)

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DFS = Discrete Fourier Series

Deals with $\tilde{x}(n)$ periodic, discrete Time signal.

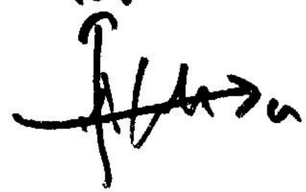
$$\tilde{x}(n) = \tilde{x}(n + kN)$$

k any integer = period.

Idea: Decompose $\tilde{x}(n)$ in Terms of exponentials. periodic with period N.

$$e_k(n) = e^{j\frac{2\pi n k}{N}}$$

$\mathrm{Re}[e_k(n)]$



$k = 0, \ldots, N-1$

There are N, periodic exponentials with period N.

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi n k}{N}}$$

weights

$e_k(n)$ is periodic with period N:

$$e_k(n) \overset{??}{=} e_{k+rN}(n)$$

r: arbitrary int.

<u>proof</u>:

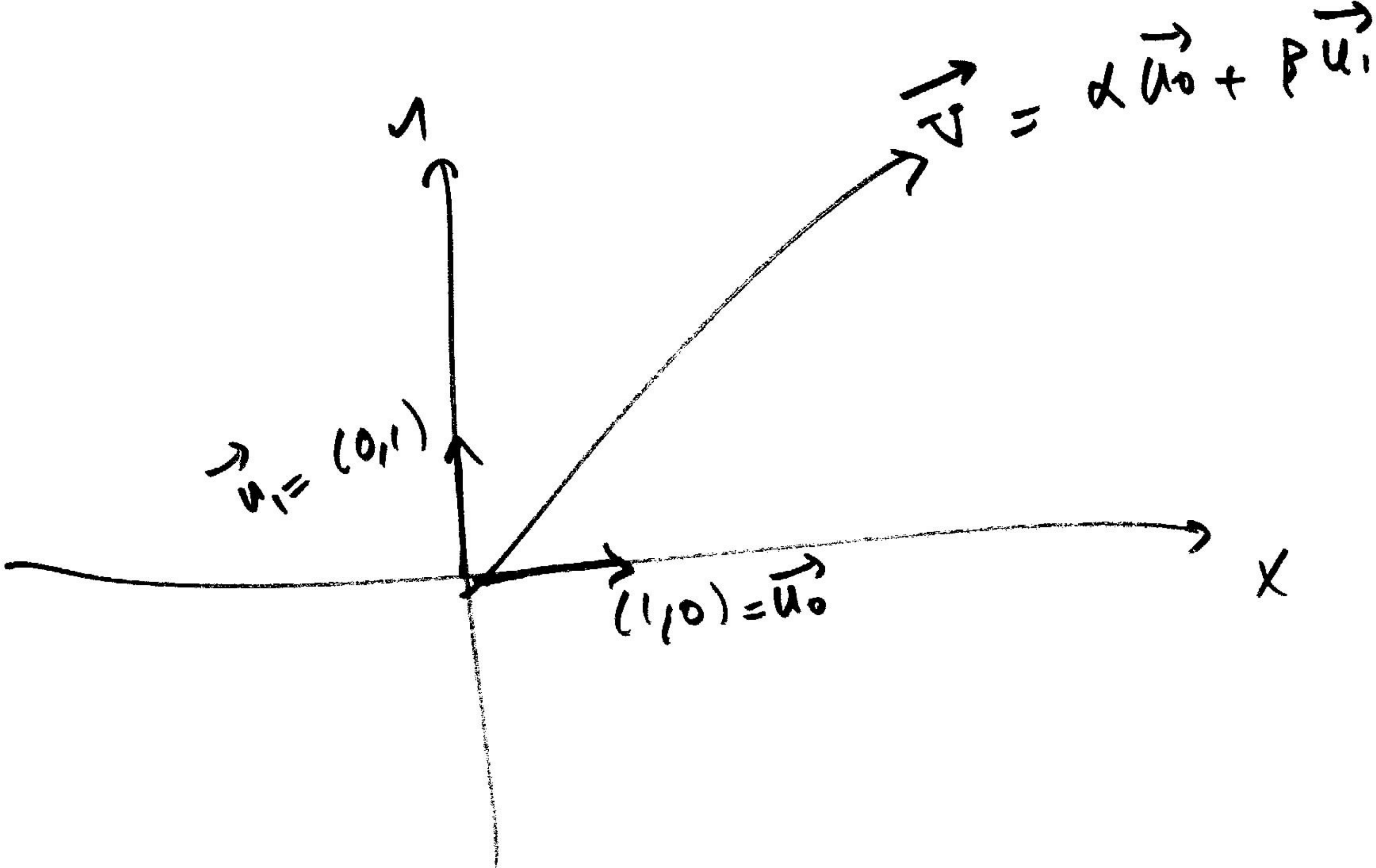
$$e^{j2\pi nk/N} \overset{?}{=} e^{j2\pi n\left(\frac{k+rN}{N}\right)}$$

$$= e^{j2\pi \frac{kn}{N}} \, e^{j2\pi n \frac{rN}{N}}$$

$$= e^{j2\pi \frac{kn}{N}}$$

$$e_0(n) = e_N(n) = e_{2N}(n) = e_{3N}(n) = \ldots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \ldots$$



$\vec{v} = \alpha \vec{u_0} + \beta \vec{u_1}$
y
$\vec{u_1} = (0,1)$
$(1,0) = \vec{u_0}$
X

Q How find "weights"? $X(k)$?

proposal:

$$X(k) \overset{??}{=} \sum_{n=0}^{N-1} \underbrace{\hat{x}(n)} e^{-j\frac{2\pi nk}{N}}$$

proof:

$$X(k) \overset{??}{=} \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{l=0}^{N-1} X(l) e^{j\frac{2\pi n l}{N}} \right) e^{-j\frac{2\pi nk}{N}}$$

$$\overset{??}{=} \sum_{l=0}^{N-1} X(l) \underbrace{\left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)n}{N}} \right]}_{A}$$

$$A = \delta(l-k-rN)$$

what is A?

$$\checkmark \overset{??}{=} \sum_{l=0}^{N-1} X(l)\,\delta(l-k-rN) = X(\underbrace{k+rN}_{r \to \text{arb. int.}}) = X(k) = X(k+N) = X(k+2N)$$

Case (1): If $l-k$ is an int. multiple of N.

$$l-k = rN$$

$$(A) = \frac{1}{N} \sum_{n=0}^{N-1} e^{j 2\pi \frac{rNn}{N}} = 1$$

Case (2) $\quad l-k \neq rN \quad$ $l-k$ is not an int. multiple of N.

$$(A) = \frac{1}{N} \sum_{n=0}^{N-1} e^{j 2\pi \frac{(l-k)n}{N}}$$

Recall $\displaystyle\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$$(A) = \frac{1}{N} \frac{1 - e^{j 2\pi \frac{(l-k)N}{N}}}{1 - e^{j 2\pi \frac{(l-k)}{N}}} = \phi$$

$$(A) = \delta(l-k-rN)$$

$\bar{X}(K) = X(K + rN)$ $\quad r =$ arb. int.

$\Rightarrow$ $X(K)$ is a periodic sequence with period N.

$\Rightarrow$ From now on refer to $X(K)$ as $\breve{X}(K)$

DFS pair.

Analysis.

$$\tilde{X}(K) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi nk}{N}}$$

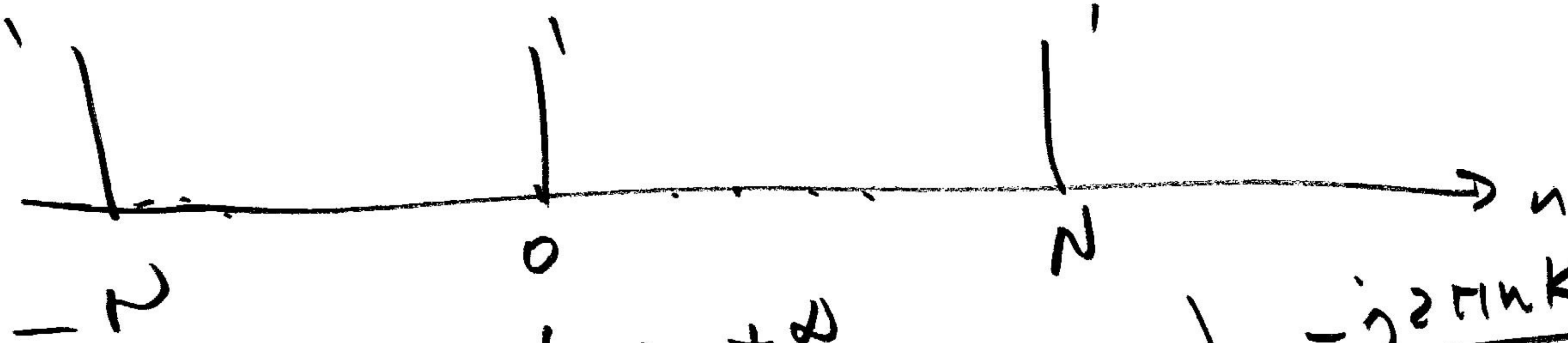
synthesis

$$\tilde{x}(n) = \frac{1}{N}\sum_{K=0}^{N-1} \tilde{X}(k)\, e^{+j\frac{2\pi nk}{N}}$$

DFS:

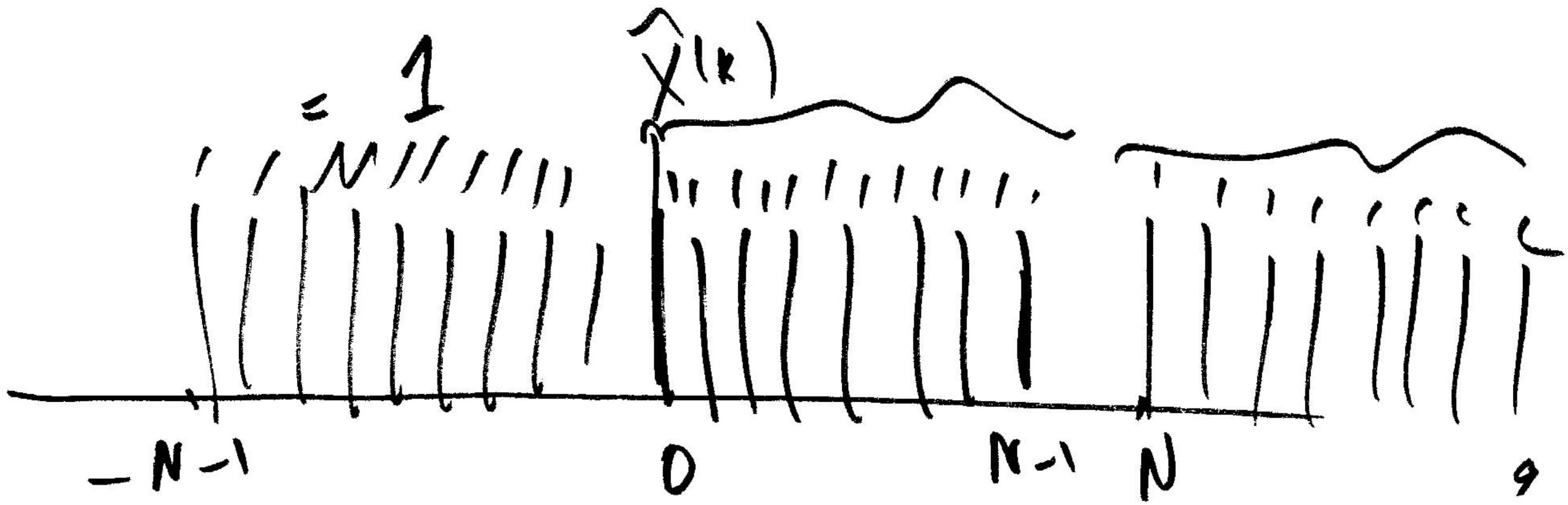
periodic N pt seq $\tilde{x}(n)$ $\longrightarrow$ periodic N point seq in freq. Domain $\tilde{X}(k)$

Ex $\tilde{x}(n) = \sum_{r=-\infty}^{\infty} \delta(n + rN)$



$$\tilde{X}(k) = \sum_{n=0}^{N-1} \left(\sum_{r=-\infty}^{+\infty} \delta(n+rN) \right) e^{-j\frac{2\pi nk}{N}}$$

$= 1$



$$\tilde{x}(n) = \boxed{\frac{1}{N} \sum_{k=0}^{N-1} e^{j\frac{2\pi nk}{N}}}$$

$$\tilde{x}(n) = \boxed{\sum_{r=-\infty}^{+\infty} \delta(n + rN)}$$

equation. train.

Shift Property

$$\tilde{x}(n) \longleftrightarrow \tilde{X}(k)$$

$$\tilde{x}(n-n_0) \longleftrightarrow e^{-j\frac{2\pi n_0 k}{N}} \tilde{X}(k)$$

Periodic Convolution:

$$\underset{\text{period } N}{\tilde{x}_1} \underset{\text{period convol}}{\circledast} \underset{\text{period } N.}{\tilde{x}_2} = \tilde{x}_3 \longrightarrow \text{period } N.$$

$$\triangleq \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)$$

$$\tilde{X}_3(k) = \tilde{X}_1(k)\, \tilde{X}_2(k)$$

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DFT = Discrete Fourier Trnsfm.

N pt

$x(n) \longrightarrow X(k)$ — N pt seq

N — DFT — N

DFS

DFT

DTFT.