Lecture: 11/7.

HW#8 due 11/19. 4.29, 4.30, 4.34, 4.36, 4.37, 4.38.

Typo for 4.37. Fig P4.37-2. $T = \frac{1}{6}\times 10^{-3}$ and not $T=\frac{1}{6}\times 10^{-4}$.

Recap: $x[n] \rightarrow \boxed{\downarrow M} \rightarrow x_d[n] = x[nM]$

$$X_d(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi k}{M}\right)$$

Define $x_k[n] = e^{j\frac{2\pi}{M}kn}\, x[n]$

let $v[n] = \frac{1}{M}\sum_{k=0}^{M-1} x_k[n] = \frac{1}{M}\sum_{k=0}^{M-1}\left(e^{j\frac{2\pi}{M}kn}\right) x[n]$

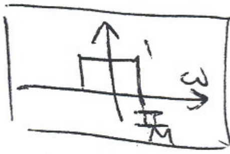
$$= \begin{cases} \frac{1}{M}\cdot M\, x[n] = x[n], & \text{if } n = Mr,\ r\in\mathbb{Z}. \\ 0, & \text{otherwise.} \end{cases}$$

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[nM]e^{-j\omega n}$$

$$= \sum_{l=-\infty}^{\infty} v[l]\, e^{-j\frac{\omega l}{M}} = V\left(\frac{\omega}{M}\right).$$

$$V(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\omega - \frac{2\pi}{M}k\right).$$

$$\Rightarrow X_d(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi}{M}k\right).$$

To avoid aliasing, $x[n] \rightarrow$  $\rightarrow \boxed{\downarrow M} \rightarrow x_d[n]$.

& $x_c(t) \rightarrow \boxed{C/D}_{\,T} \rightarrow x[n] \rightarrow \boxed{\downarrow M} \rightarrow x_d[n]$

& $x_c(t) \rightarrow \boxed{C/D}_{\,MT} \rightarrow x_d[n]$

} if $\Omega_N \le \frac{\pi}{MT}$, then $x_c(t)$ can be recovered.

Upsampling.

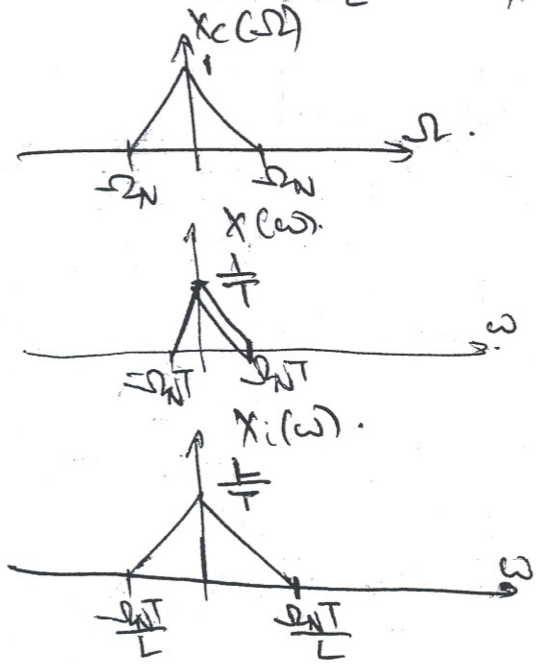
Motivation: $x_c(t) \to \boxed{C/D} \to x[n]$, $x[n] = x_c(nT)$

T

wish to obtain $x_c(t) \to \boxed{C/D} \to x_i[n]$, $x_i[n] = x_c(nT/L)$

T/L , $L \in \mathbb{Z}$.

Eg.



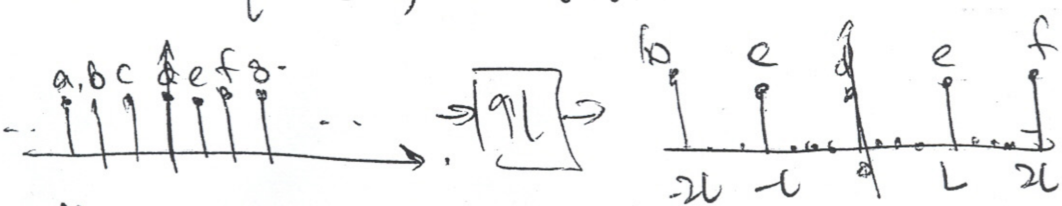
choose $\Omega_N = \frac{\pi}{T}$.

Consider this system.

$$x[n] \to \boxed{\uparrow L} \xrightarrow{x_e[n]} \underset{h_i[n]}{\boxed{LPF}} \to x_i[n].$$

$$\text{where } x_e[n] = \begin{cases} x[n/L], & \text{if } n = rL,\ r \in \mathbb{Z}. \quad (0, \pm L, \pm 2L, \dots) \\ 0, & \text{otherwise.} \end{cases}$$

eg

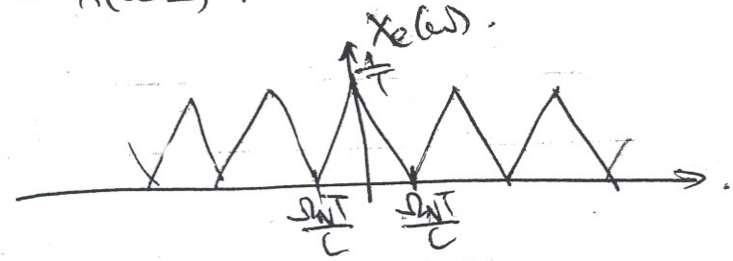


$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL].$$

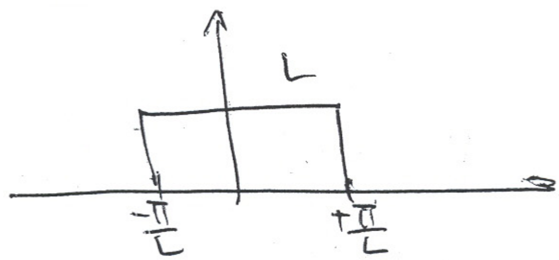
$$X_e(\omega) = \sum_{n=-\infty}^{\infty} x_e[n]\, e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x_e[rL]\, e^{-j\omega rL} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega rL}$$

$$= X(\omega L).$$

i.e.



Hence, to get $X_i(\omega)$, we need a low pass filter of:



Now, $h_i[n] = \dfrac{\sin(\pi n/L)}{\pi n/L}$. → notice that $h_i[n] = 0 \;\forall\; n = \pm L, \pm 2L$ & $h_i[0] = 1$.

$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n - kL]$

$x_i[n] = x_e[n] * h_i[n] = \sum_{k=-\infty}^{\infty} x[k]\, \dfrac{\sin(\pi(n-kL)/L)}{\pi(n-kL)/L}$.

Hence, $x_i[n] = x[n/L] = x_c(nT/L)$ for $n = 0, \pm L, \dots$

Changing sampling rate by non-integer:

~~Continuous time interpolation.~~

Suppose $x_c(t) \rightarrow \boxed{C/D} \rightarrow x[n]$, with sampling period T.

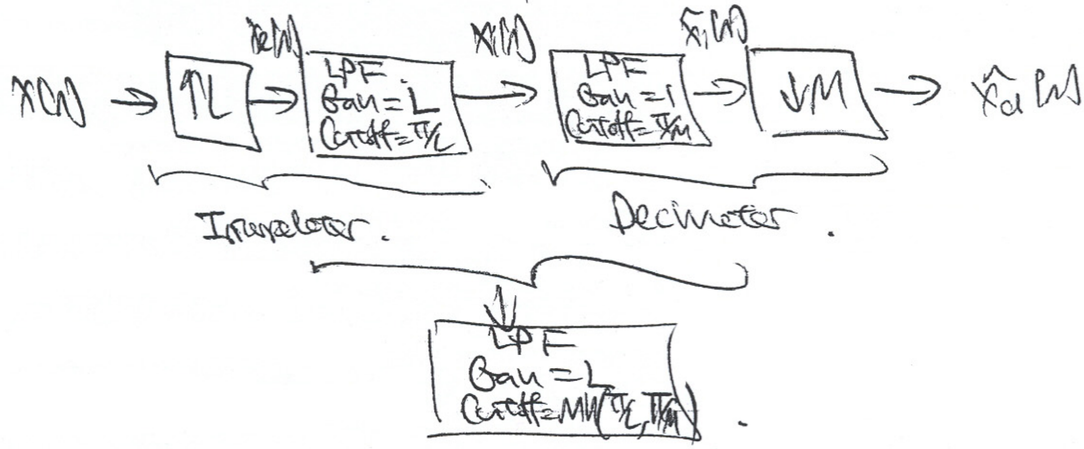
& $x_c(t) \rightarrow \boxed{C/D} \rightarrow \tilde{x}_d[n]$, with sampling period $\frac{M}{L}T$. , $M, L \in \mathbb{Z}$.

i.e. downsample by M & upsample by L.

$M > L \Rightarrow$ net downsampling.

$M < L \Rightarrow$ net upsampling.

To implement w/o aliasing, need.



Example.

Let $T' = \frac{3}{2}T$, with $\Omega_N = \frac{\pi}{T'} = \frac{\pi}{\frac{3}{2}T} = \frac{2}{3}\frac{\pi}{T}$.

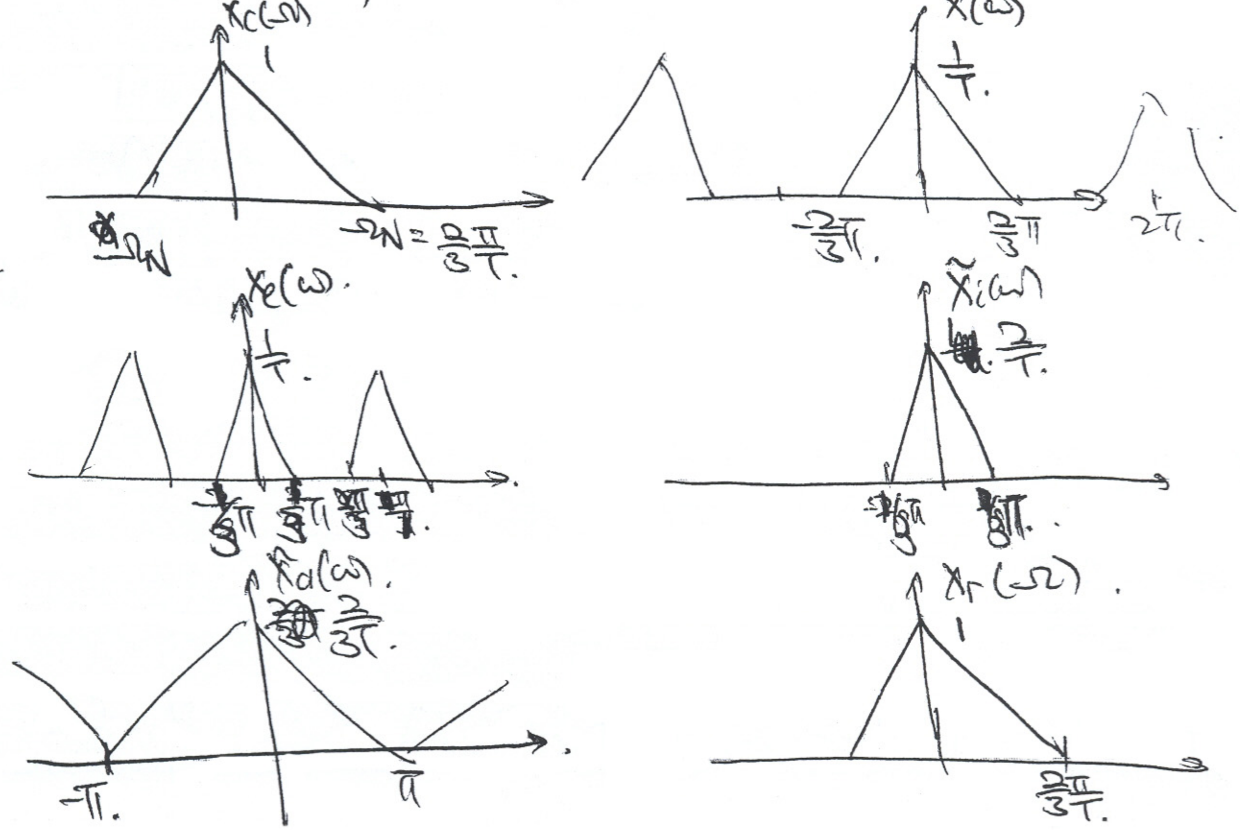
L = 2

M = 3.

$\frac{2\pi}{3} = \frac{6\pi}{9}$

$\frac{2}{3T} \cdot \frac{3}{2}T$

$\frac{\pi}{\frac{3}{2}T} = \frac{2}{3}$



Interchange of down-sampling & filtering.

$$x[n] \rightarrow \boxed{\downarrow M} \xrightarrow{x_a[n]} \boxed{H(z)} \rightarrow y[n]$$

$$\equiv \; x[n] \rightarrow \boxed{H(z^M)} \xrightarrow{x_b[n]} \boxed{\downarrow M} \rightarrow y[n].$$

$$X_b(\omega) = H(\omega M) X(\omega).$$

$$\& \; Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X_b\left(\frac{\omega}{M} - \frac{2\pi k}{M}\right) = \frac{1}{M}\sum_{k=0}^{M-1} H(\omega - 2\pi k) X\left(\frac{\omega}{M} - \frac{2\pi k}{M}\right)$$

$$= H(\omega) \cdot \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega}{M} - \frac{2\pi k}{M}\right)$$

$$= H(\omega) X_a(\omega).$$

Also,

$$x[n] \rightarrow \boxed{H(z)} \xrightarrow{x_a[n]} \boxed{\uparrow L} \rightarrow y[n]$$

$$\equiv \; x[n] \rightarrow \boxed{\uparrow L} \xrightarrow{x_b[n]} \boxed{H(z^L)} \rightarrow y[n].$$

$$Y(\omega) = X_a(\omega L) = H(\omega L) X(\omega L).$$

Also, $X_b(\omega) = X(\omega L) \Rightarrow Y(\omega) = H(\omega L) X_b(\omega)$ ← system (b).