Linear Phase Filtering

$$e^{j\omega n} \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow e^{j\omega n} \underbrace{H(\omega)}_{\downarrow}$$

$DTFT\{h(n)\}$

$=$ Freq. Response filter at ω.

$$x(n) \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y(n)$$

$$Y(\omega) = X(\omega)\, H(\omega)$$

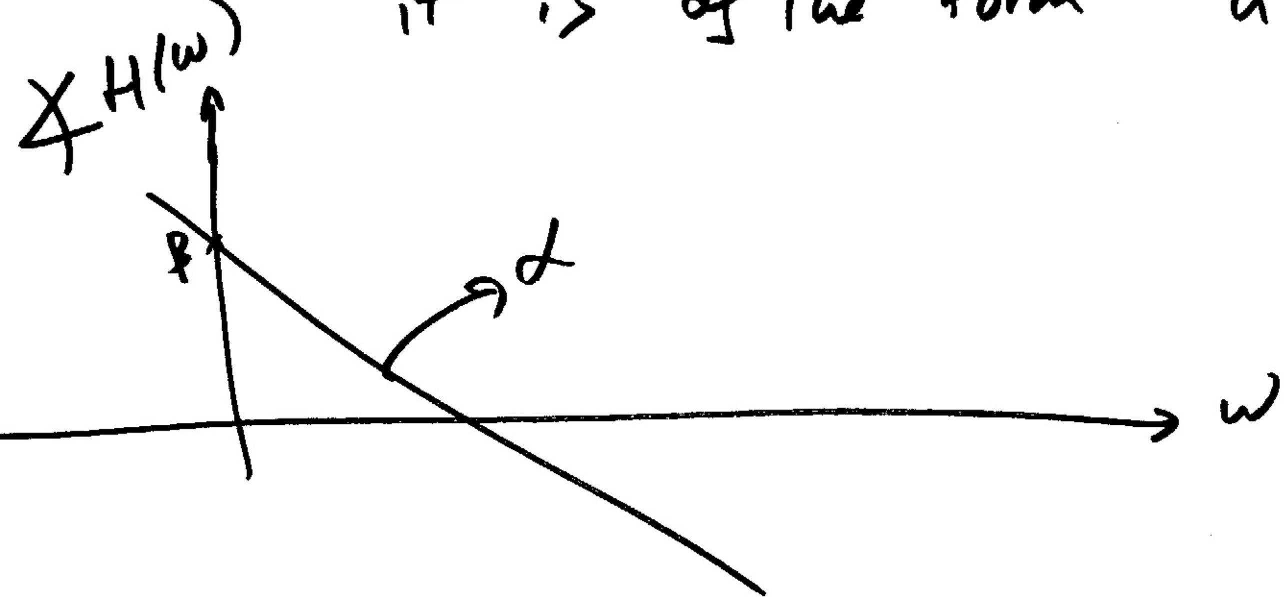
$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

Def Linear phase $\underbrace{\text{LTI system}}_{\text{filter}}$:

if $\measuredangle H(\omega)$ is linear in ω

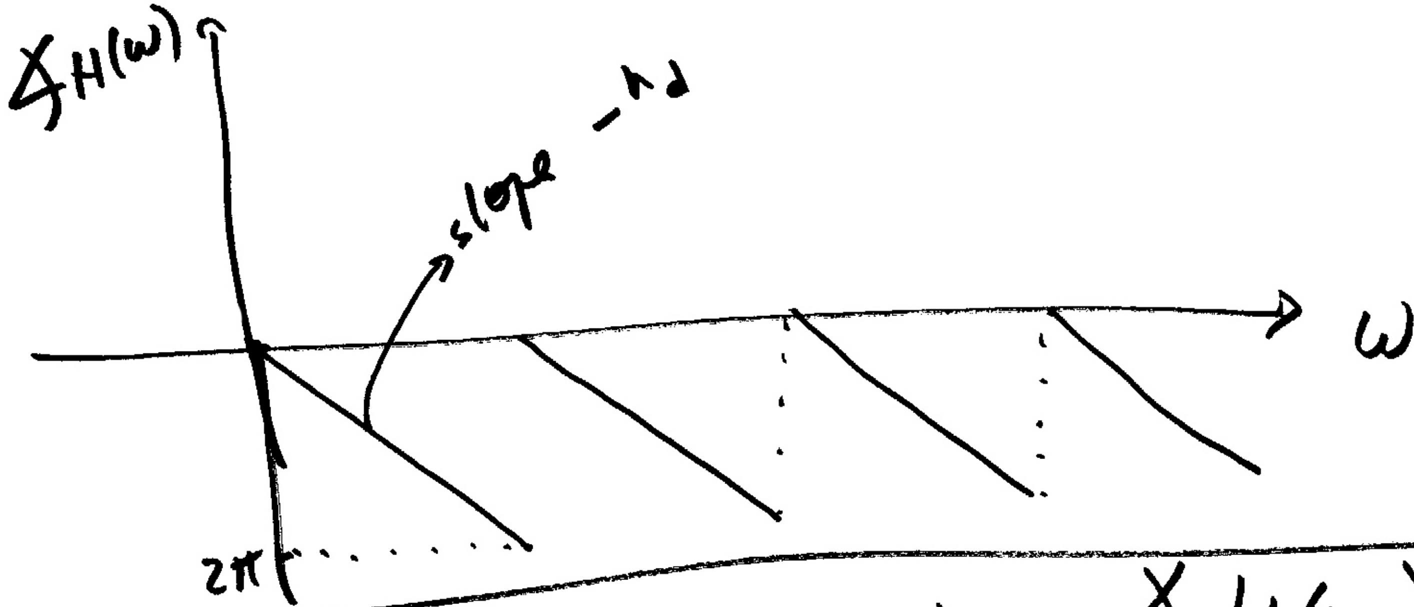
it is of the form $\alpha\omega + \beta$



Consider a pure delay LTI system

$$h(n) = \delta(n - n_d)$$

$|H(\omega)| = 1$ $\measuredangle H(\omega) = -\omega n_d$



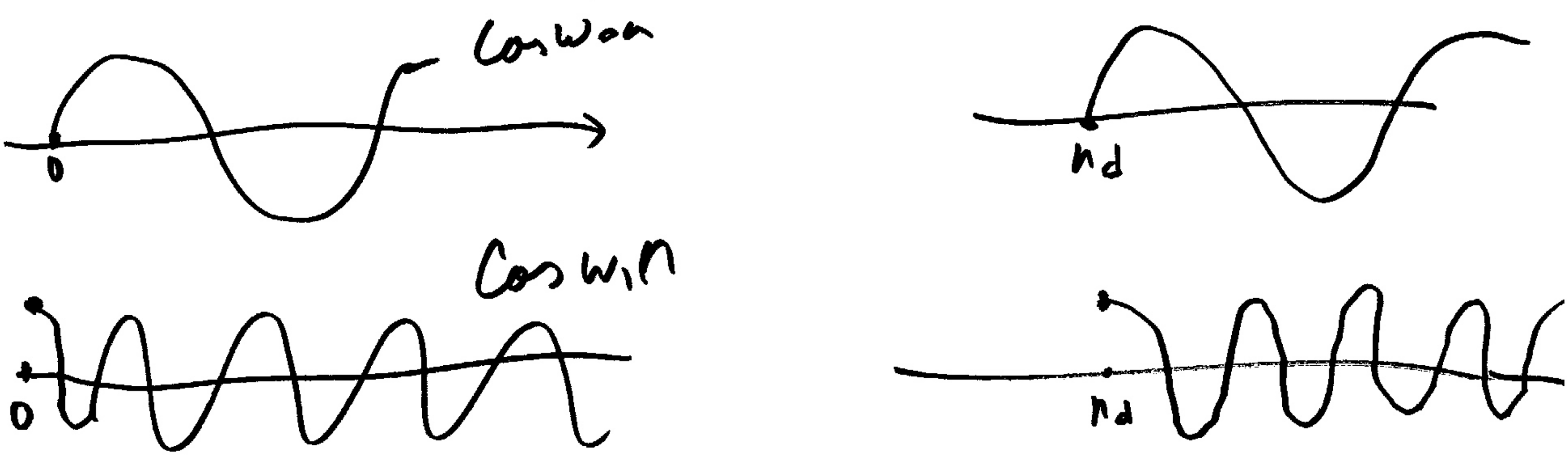
Def. $\boxed{\text{Group delay} \triangleq -\frac{d}{d\omega} \measuredangle H(\omega)}$

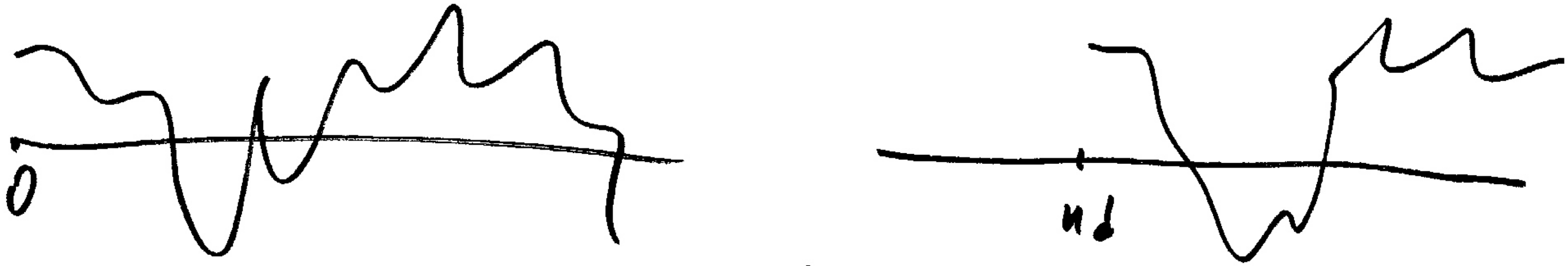
$\measuredangle H(\omega) = -\omega n_d \Rightarrow \text{G.D.} = -\frac{d}{d\omega}(-\omega n_d) = n_d$

intuitively. n_d is the amount of delay for <u>ALL</u> freq. is indpendent of $\underline{\omega}$

All frequencies are delayed by the same amount as they pass through the system.

$\cos \omega_0 n + \cos \omega_1 n + \longrightarrow$ $\boxed{\begin{array}{l} h(n) \\ \text{G.D. } n_d. \\ h(n) = \delta(n - n_d) \end{array}}$





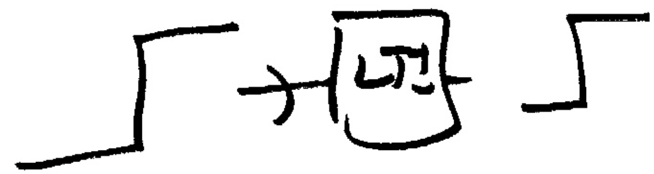
<u>observation</u> is Linear phase $\longrightarrow$ group delay is constant.

$\Rightarrow$ relative phase between sinusoidal components are same as they go through the system

$\Rightarrow$ no distortion added to the signal

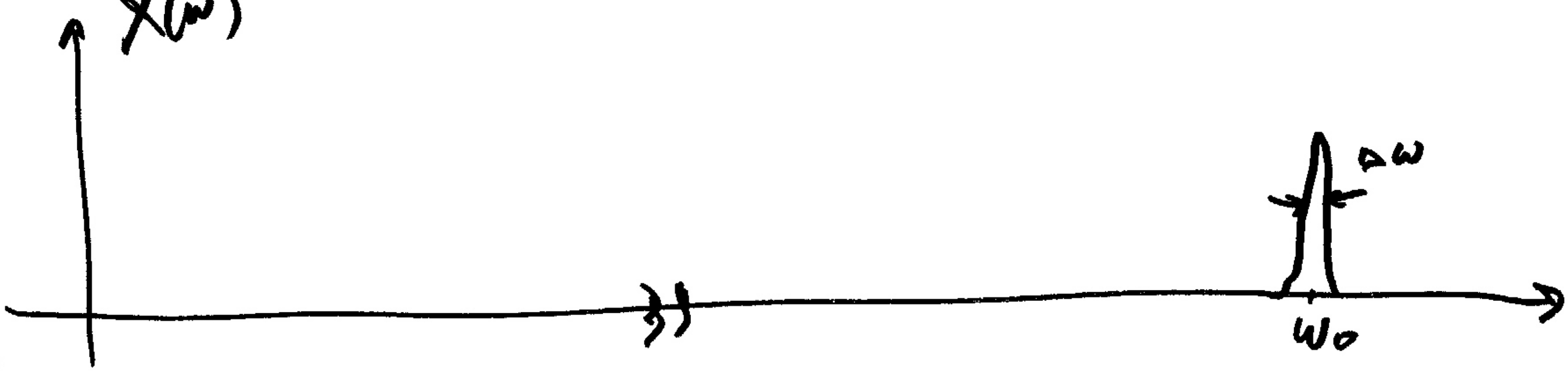
<u>Q</u> why preserving relative phase is important?

image processing.



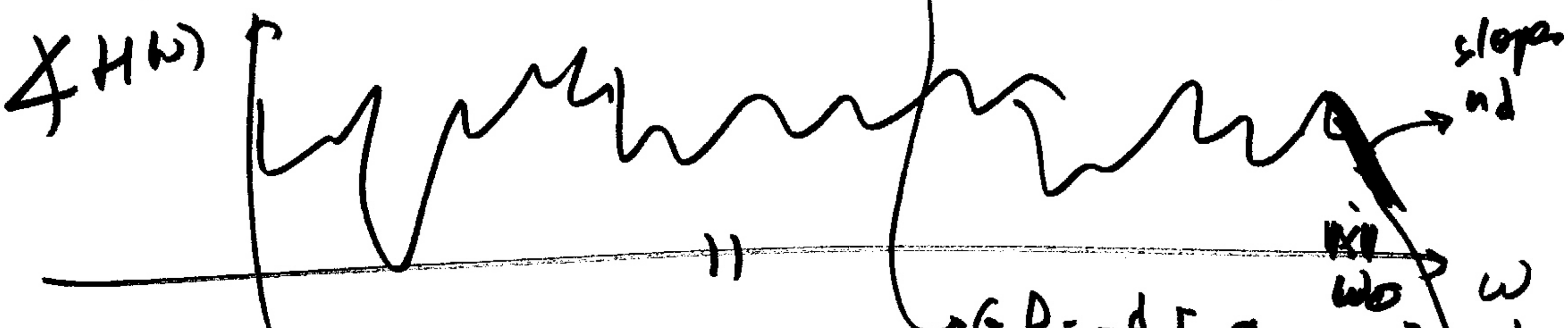
To preserve edges in image processing, it is good to have linear phase

Consider a narrow band signal. cellphone $\omega_0 = 900\,MHz$.
$\Delta\omega = 5\,KHz$



If $x(n) = s(n)\cos\omega_0 n$

- pass $x(n)$ through LTI filter $H(\omega)$. s.th.
- $|H(\omega)| = 1$, can approximate $\measuredangle H(\omega)$ at $\omega = \omega_0$ with a linear term $\left[\measuredangle H(\omega)\right]_{\omega=\omega_0} = -\phi - \omega n_d$



$G.D. = -\frac{d}{d\omega}[-\phi - \omega n_d] = n_d$

Then $\quad x(n) \rightarrow \boxed{H(\omega)} \rightarrow y(n)$

Can show: $\quad y(n) = s(n - n_d)\cos(\omega_0 n - \omega_0 n_d - \phi)$

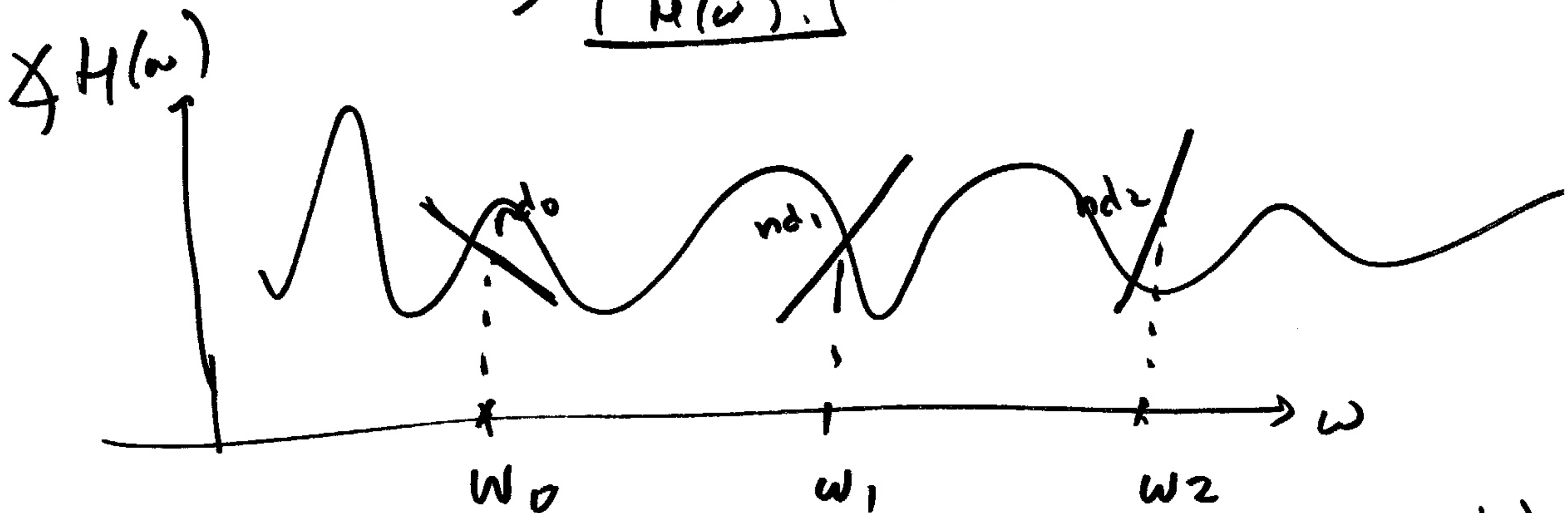
Conclusion: For a narrowband signal centered around ω_0 delay is proportional To

$$GD = \left[-\frac{d}{d\omega} \measuredangle H(\omega)\right]_{\omega=\omega_0}$$

ω_0 = is center freq. of narrowband signal.

Consider $\quad x(n) = a\cos\omega_0 n + b\cos\omega_1 n + c\cos\omega_2 n.$

$$x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ H(\omega) \end{matrix}} \longrightarrow y(n)$$

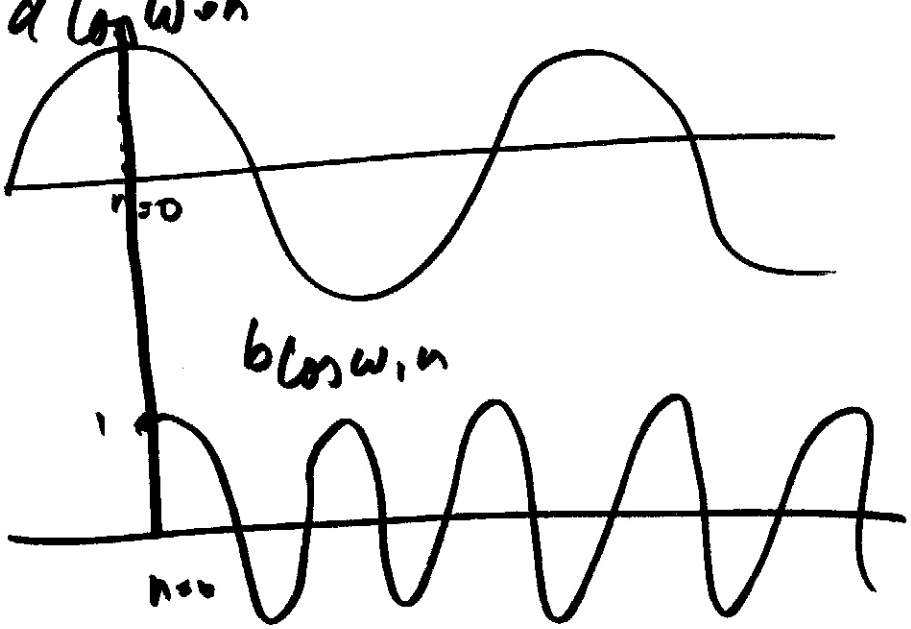


$$a\cos\omega_0 n \longrightarrow a\cos(\omega_0 n - \omega_0 n_{d_0})$$

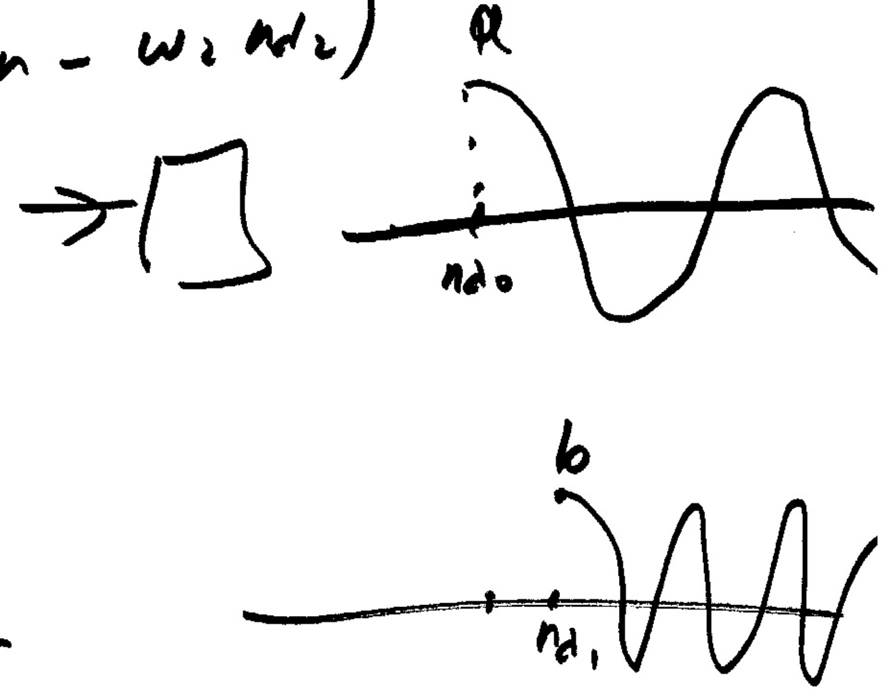
$$b\cos\omega_1 n \longrightarrow b\cos(\omega_0 n - \omega_1 n_{d_1})$$

$$c\cos\omega_2 n \longrightarrow c\cos(\omega_2 n - \omega_2 n_{d_2})$$

$$y(n) = a\cos(\omega_0 n - \omega_0 n_{d_0}) + b\cos(\omega_1 n - \omega_1 n_{d_1}) + c\cos(\omega_2 n - \omega_2 n_{d_2})$$



$n = n_0$
$a+b$ = amplitude



Unless $n_{d_0} = n_{d_1}$
$\Rightarrow$ introduced distortion

Linear Phase filtering

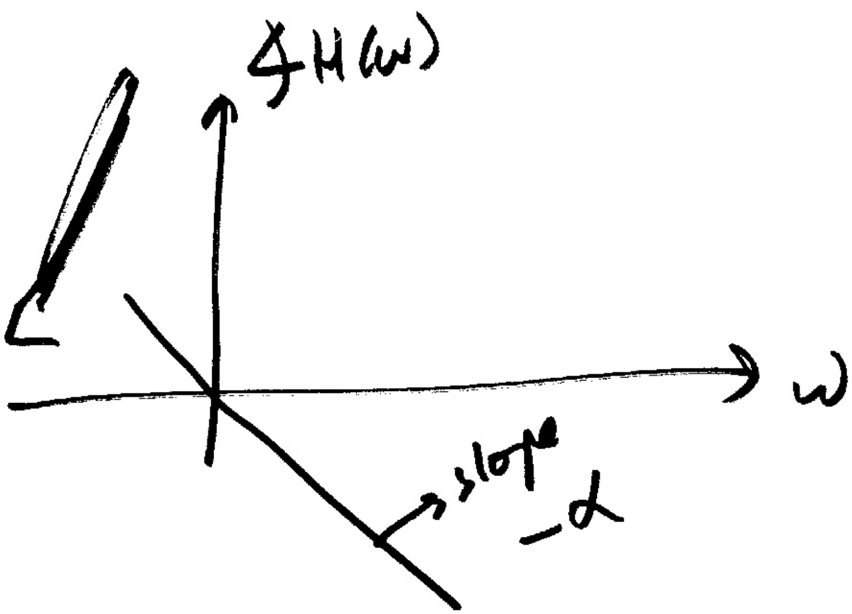
Def

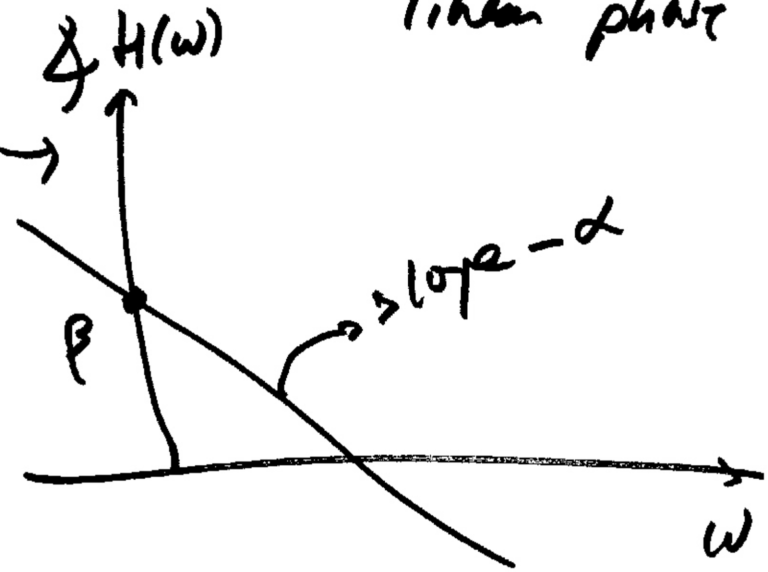
$$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real.}} e^{-j\alpha\omega}$$

linear phase

$$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{j(\beta - \alpha\omega)}$$

Generalized linear phase



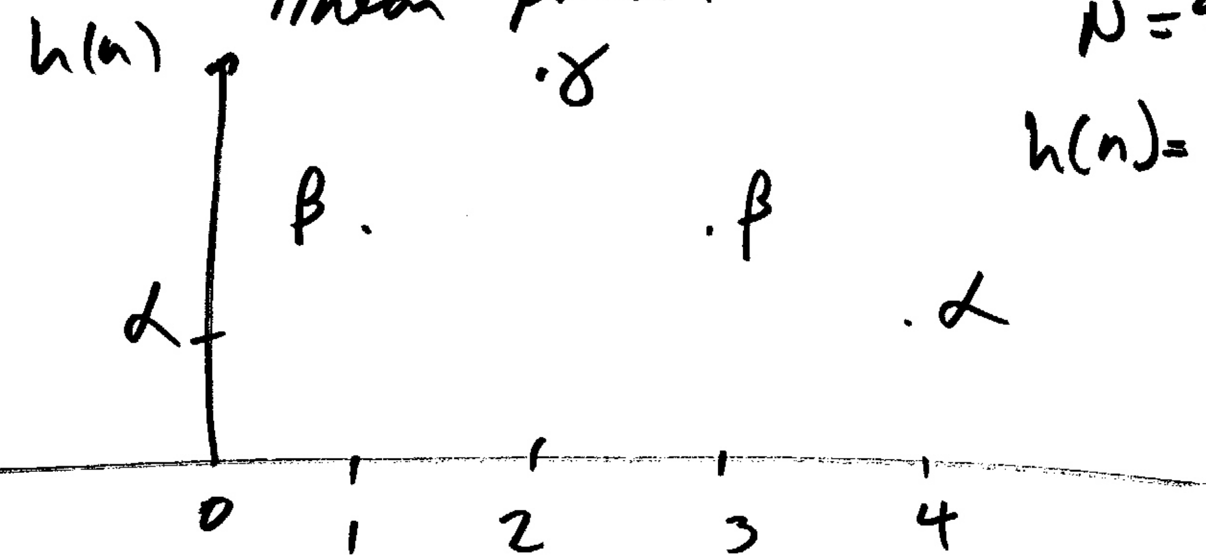


Theme: How to get filters that are linear phase? (design)

Answer: Impose symmetry or anti symmetry on filter coeff.

Show: If $h(n) = h(N-1-n)$ for FIR filter with real coeff $\Longrightarrow$ Then linear phase.

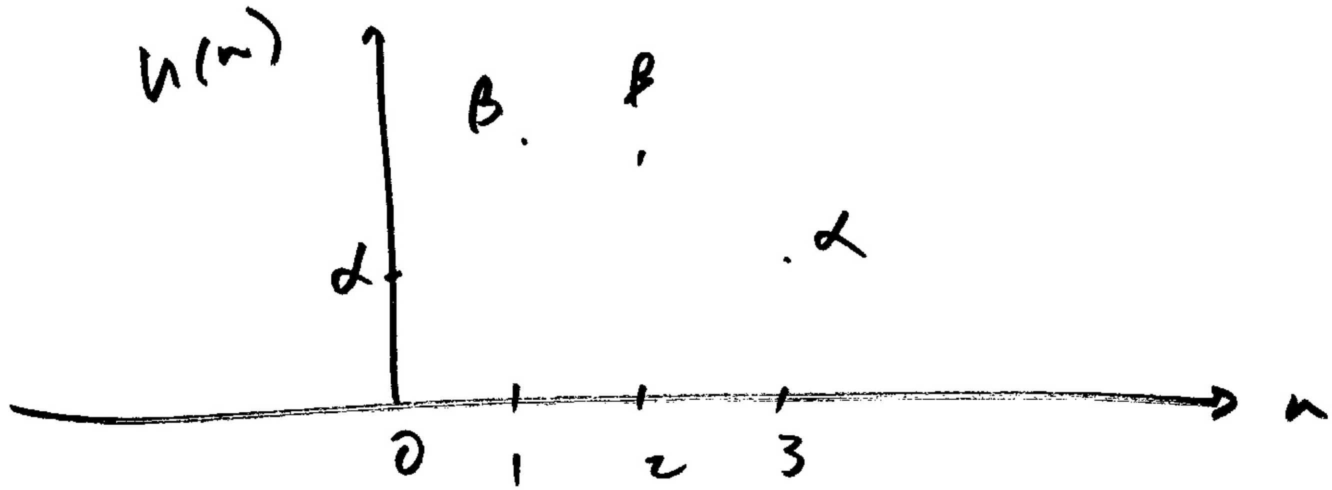
EX



$N = 5$

$h(n) = h(4-n)$

Ex $N=4$ $h(n) = h(3-n)$



Proof: $H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \underbrace{\sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}}$$

change of variable $m = N-1-n$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) e^{-j\omega n} + \sum_{m=0}^{N/2-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

impose assumption $h(n) = h(N-1-n)$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} \cdot e^{+j\omega\left(\frac{N-1}{2}\right)} + e^{j\omega n} e^{-j\omega\left(\frac{N-1}{2}\right)} \right]$$

$H(\omega) =$

$$H(\omega) = \underbrace{e^{-j\omega\left(\frac{N-1}{2}\right)}}_{e^{-j\alpha\omega}} \underbrace{\sum_{n=0}^{N/2-1} h(n)\left[2\cos\left(\omega n - \frac{\omega}{2}(N-1)\right)\right]}_{\substack{\text{Real} \\ H_m(\omega)}}$$

with $\alpha = \frac{N-1}{2}$

$$H(\omega) = H_m(\omega)\, e^{-j\alpha\omega}$$

$\Longrightarrow$ linear phase.

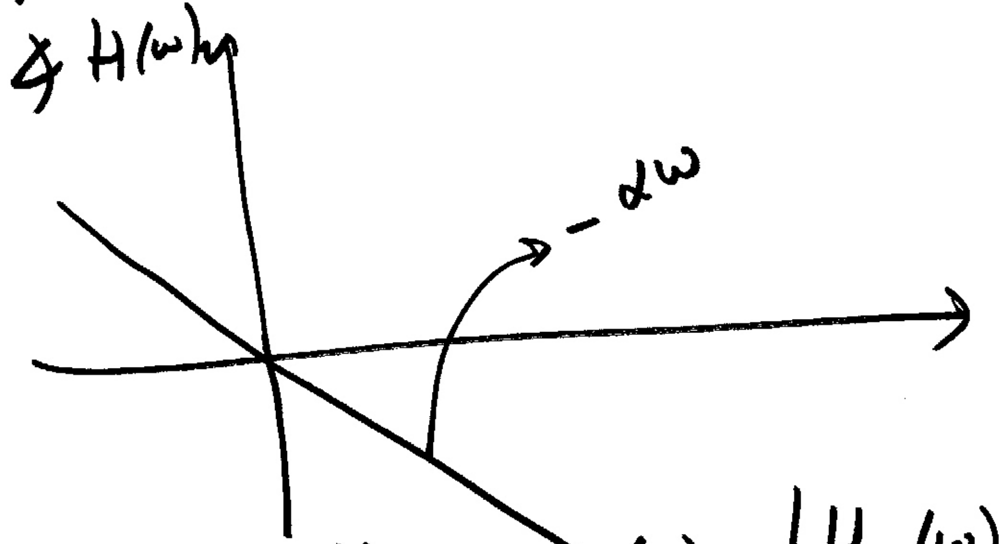
Q: what is $\measuredangle H(\omega)$

$$H(\omega) = |H(\omega)|\, e^{j\measuredangle H(\omega)} = H_m(\omega)\, e^{-j\alpha\omega}$$

Consider 2 case

① $H_m(\omega) > 0$ positive $\Longrightarrow \measuredangle H(\omega) = e^{-j\alpha\omega}$

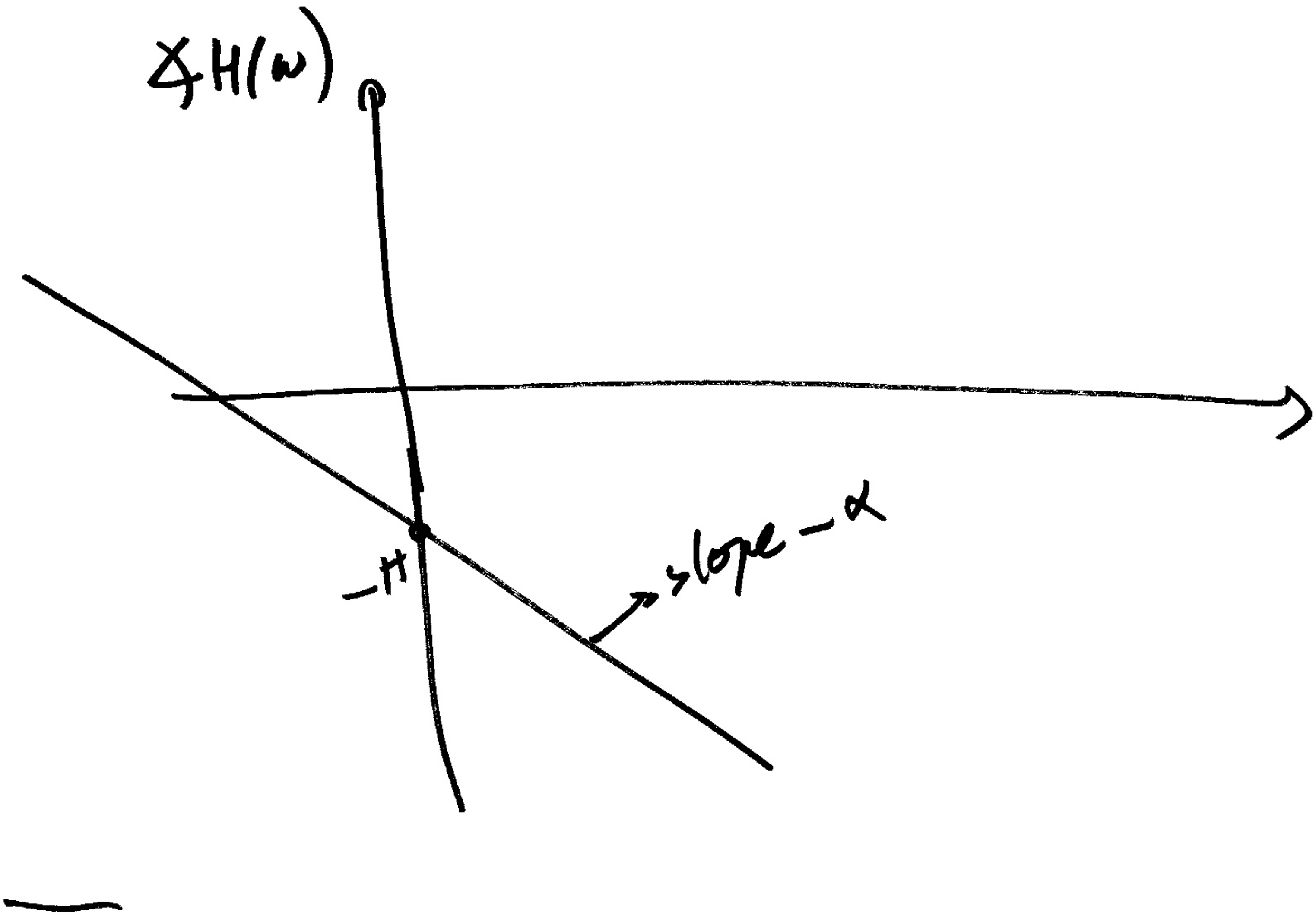
$\Longrightarrow |H_m(\omega)| = |H(\omega)|$



② $H_m(\omega) < 0$ negative $\Longrightarrow H(\omega) = |H_m(\omega)|(-1)\, e^{-j\alpha\omega}$

$$= |H_m(\omega)|\, e^{-j\pi}\, e^{-j\alpha\omega}$$

$$= |H_m(\omega)|\, e^{-j(\alpha\omega + \pi)}$$


$\measuredangle H(\omega)$
$-H$
slope $-\alpha$

<u>Show</u> symmetry condition that makes FIR filters linear phase, also reduces implementation complexity

$h(n) = h(N-1-n)$

$$x(n) \rightarrow \boxed{\begin{matrix} h(n) \\ \text{FIR} \end{matrix}} \rightarrow y(n).$$

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

$$= \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{k=N/2}^{N-1} h(k)\, x(n-k)$$

change of variable

$k = N-1-m$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{m=0}^{N/2-1} h(N-1-m)\, x(n-N+1+m)$$

$h(m)$

$$y(n) = \sum_{k=0}^{N/2-1} h(k) \underbrace{\left[x(n-k) + x(n-N+1+k)\right]}_{W(K)}$$

Each $y(n)$ $\left.\begin{matrix} N/2 \text{ mults} \\ N \text{ adds} \end{matrix}\right\} \longrightarrow$ linear phase

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$$\sum_{k=0}^{N-1} h(k)\, x(n-k)$$

N mults

N adds.

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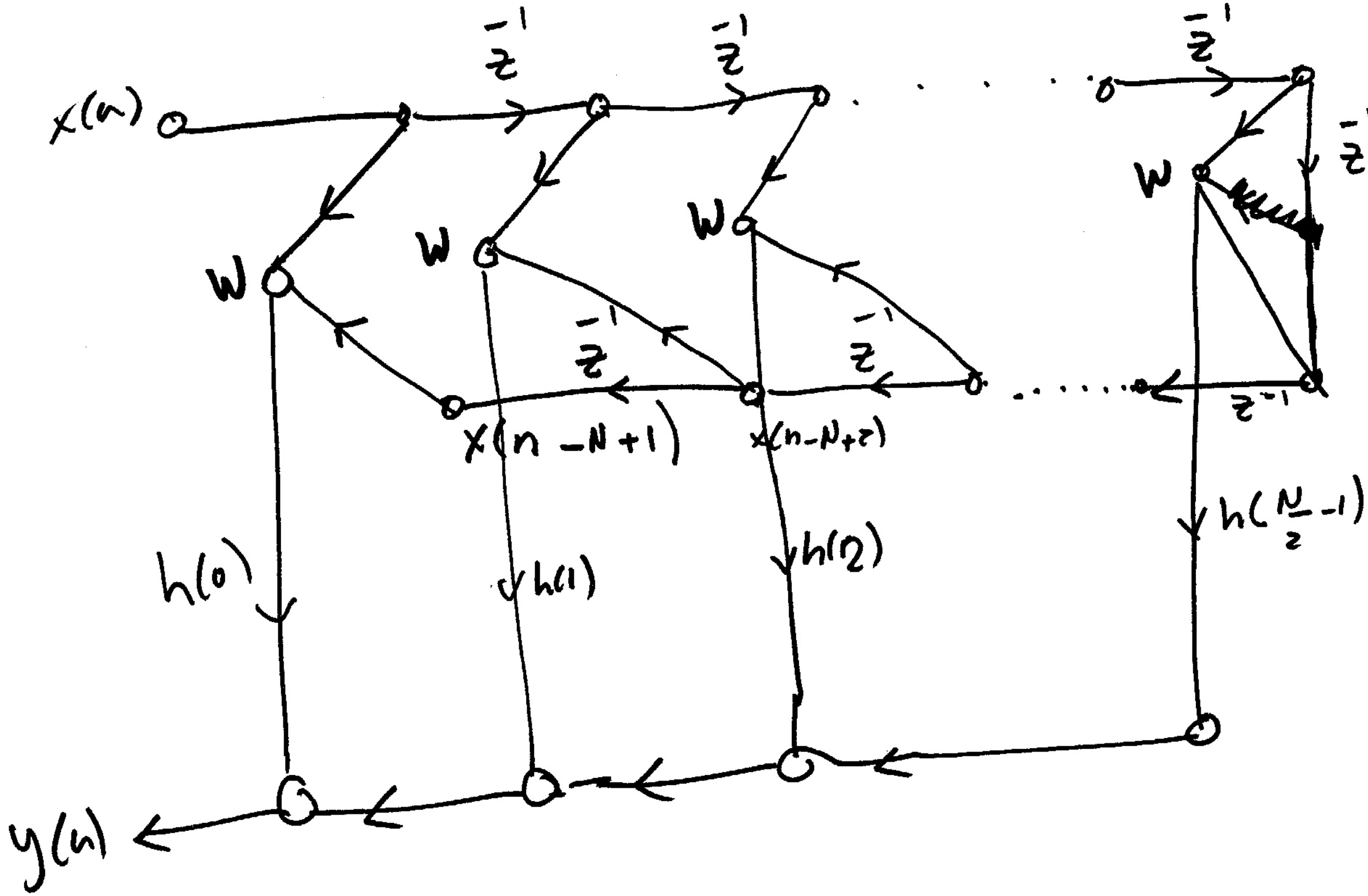
linear phase $\quad$ 1/2 as many mult.



x(n)
z⁻¹
z⁻¹
z⁻¹
z⁻¹
W
W
W
W
z⁻¹
z⁻¹
z⁻¹
x(n-N+1)
x(n-N+2)
h(0)
h(1)
h(2)
h(N/2-1)
y(n)