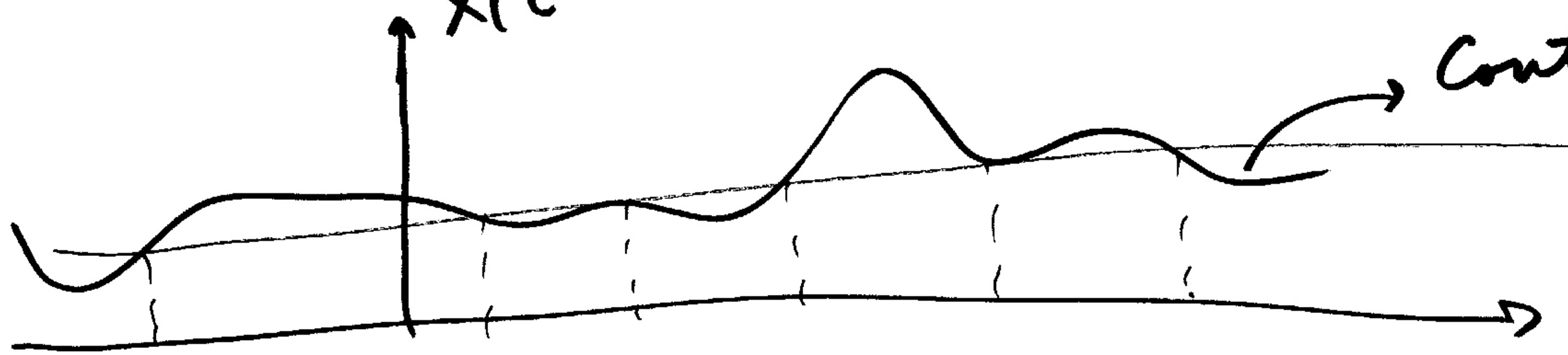
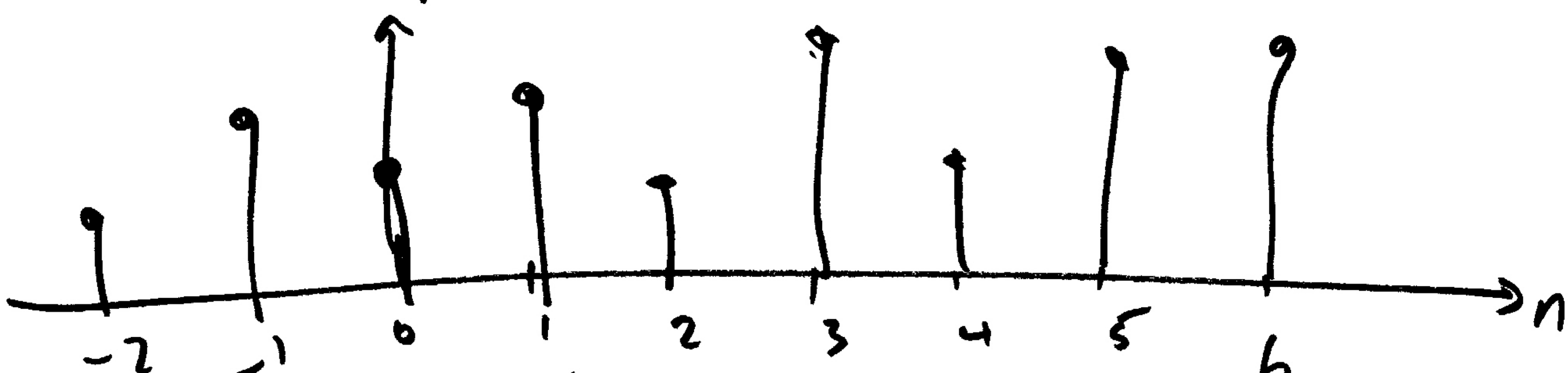
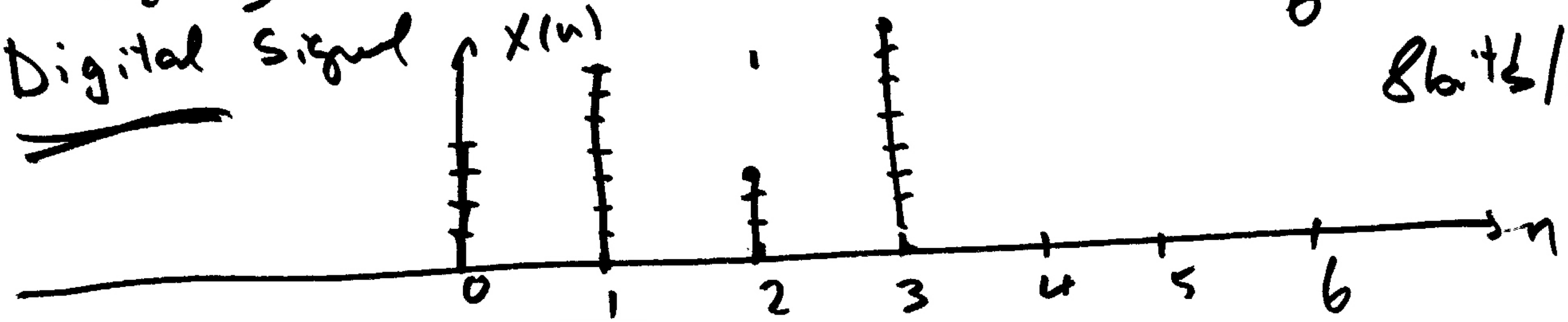
Digital Signal Processing (DSP)



Discretize C.T. signal . in Time



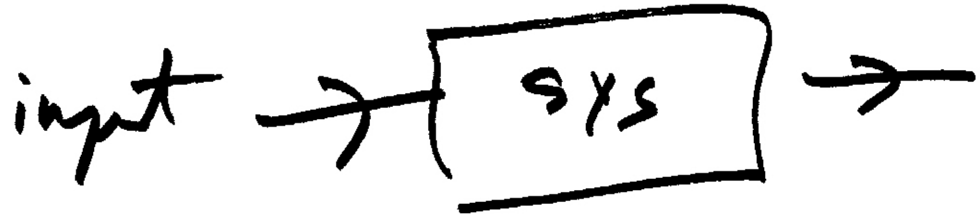
Digital Signal



8bits/16bits

System

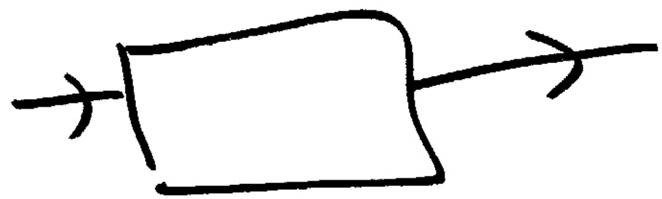
Processes a signal.



$x(n)$ → $y(n) = T\{x(n)\}$

unique input result in a unique output.

Linear Systems:



$x(n)$ → $y(n) = T\{x(n)\}$

Linear.

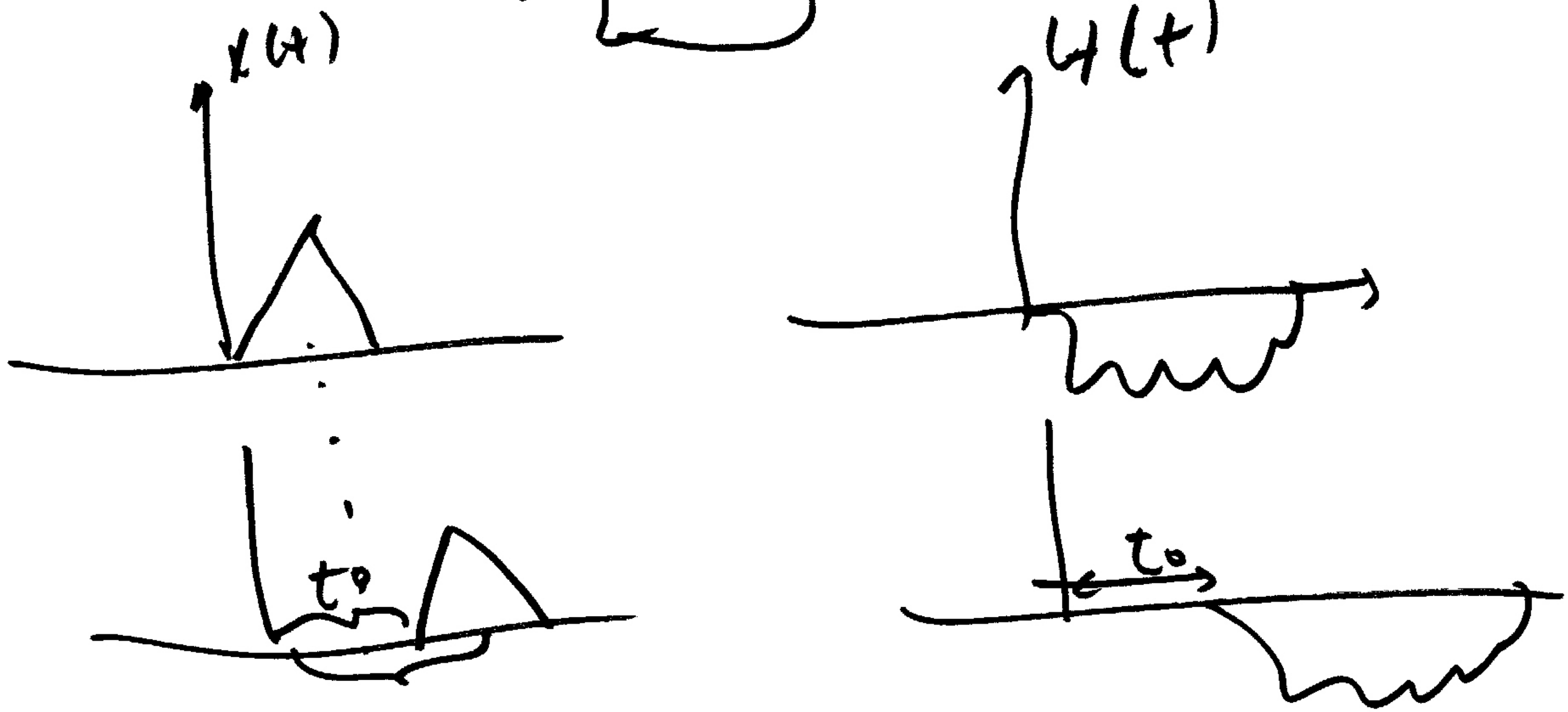
$$T\{a\,x(n)\} = a\,T\{x(n)\} = y(n)$$

$x_1(n) \to y_1(n)$; $x_2(n) \to y_2(n)$ $\quad x_1(n) + x_2(n) \to y_1(n) + y_2(n)$

Time Invariance

$x(t) \longrightarrow \boxed{} \longrightarrow y(t).$

$x(t-t_0) \longrightarrow \boxed{} \longrightarrow y(t-t_0)$



$x(t) \longrightarrow \boxed{} \longrightarrow \underline{t}\ x(t) = y(t)$

Linearity + T.I. = LTI.

LTI →: <u>Impulse response</u> completely characterizes it.

$\delta(n) \longrightarrow \boxed{LTI} \longrightarrow$ impulse response $= h(n)$

$$x(n) \longrightarrow \underset{LTI}{\boxed{h(n)}} \longrightarrow y(n) = x * h.$$

$$y(n) = \sum_{k} h(k)\, x(n-k)$$

Convolution

$h(n)$: ① of finite duration

finitely many nonzero values

FIR = Finite Impulse Respon

② ∞'ly many nonzero values.

IIR = Infinite Impulse Respon

Rational Transfer fn. = Ratio of polynomials

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{P(z)}{Q(z)} \longrightarrow$ Diff. Equation To implement this.

polynomial in z.

IIR.

Not a Rational Transfer function

$h(n)$


FIR
IIR
Rational Transfer fns

LTI

Causality:

LTI is **causal** $\longleftrightarrow$ $h(n) = 0 \quad n < 0$

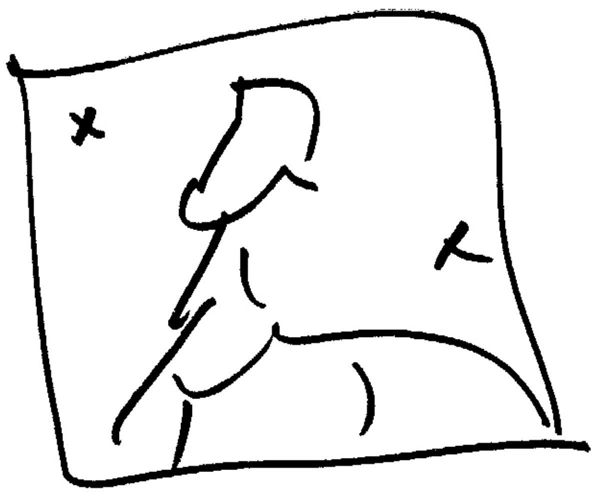


image.

Stability

BIBO = Bounded input Bounded output.

- LTI systems BIBO stable $\longleftrightarrow$ absolutely sumable $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

Eigen function

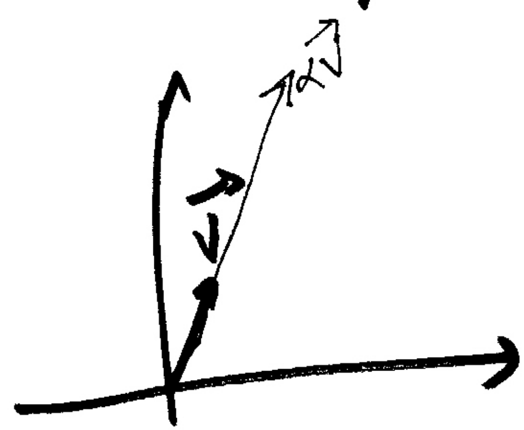
eigen-value / eigen vector.

$A, \vec{v}$:

$A\vec{v} = \alpha\vec{v}$

$A = [\vdots\ \vdots]$.

If $\vec{v}$ is eigenvector
of matrix A,
then applying
matrix to $\vec{v}$,
doesnot change direction of it, only the magnitude.



Eige

Eigenfunction

$$\phi(n) \longrightarrow \boxed{\text{System}} \longrightarrow \alpha\,\phi(n)$$

Show for LTI systems. $e^{j\omega n}$ = complex exponential.

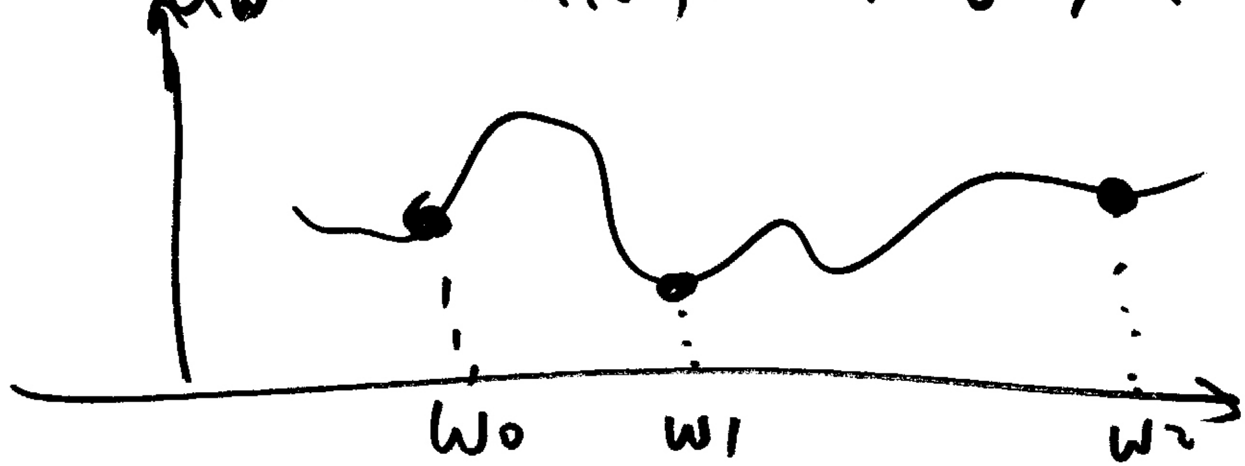
is eigen function.

$$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow \alpha\, e^{j\omega n}.$$

α → function of ω.

$\alpha(\omega) = \alpha_\omega$

$\alpha_\omega \cdot \alpha(\omega)$, $H(\omega)$ = Frequency Respon



$$x(n) = e^{jwn} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(n)\end{matrix}} \longrightarrow y(n)$$

$$y(n) = \sum_k e^{jwn}\, h(n-k) = \sum_m e^{jw(n-m)}\, h(m)$$

$$= e^{jwn} \underbrace{\sum_m e^{-jwm}\, h(m)}_{\alpha_w = \alpha(\omega) = H(\omega)}$$

DTFT of $h(n)$

Frequency Response for LTI system

$$\boxed{\text{DTFT}\{h(n)\} = H(\omega) = \sum_n h(n)\, e^{-jwn}}$$

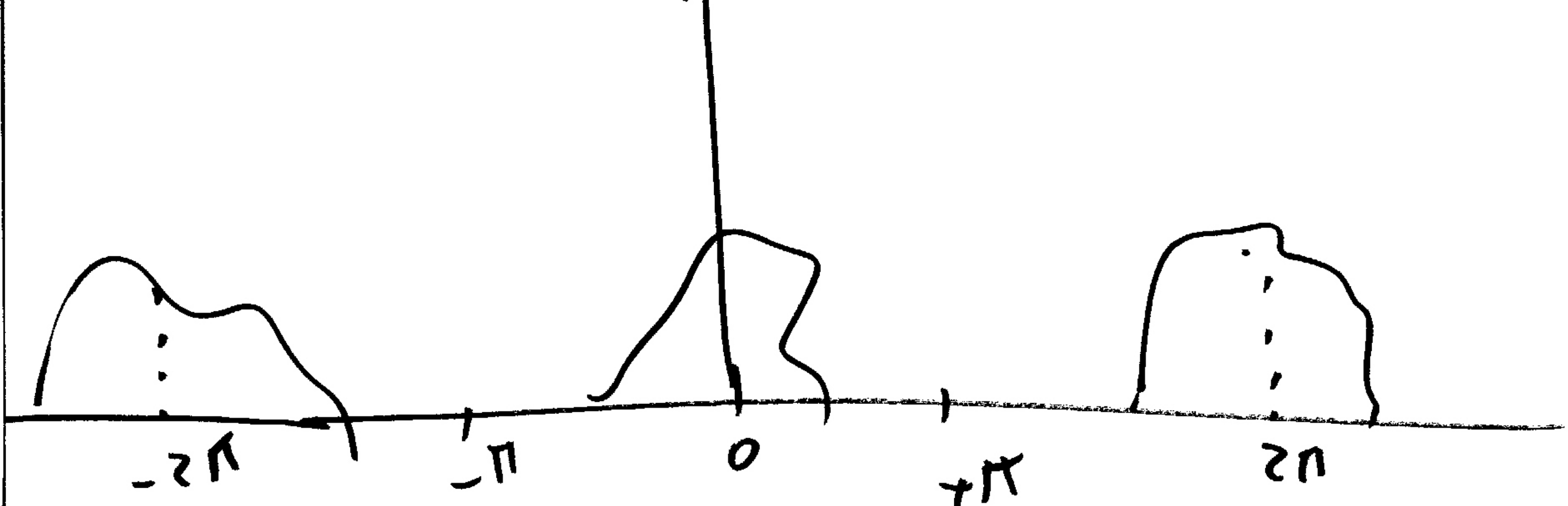
$x(n)$:

$$\mathrm{DTFT}\{x(n)\} = X(\omega) = \sum_n x(n) e^{-j\omega n}$$

n = integer. ω = continuous.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega) e^{j\omega n} d\omega$$

$X(\omega) \longrightarrow$ periodic with period 2π



$$x \longrightarrow \boxed{h(n)} \longrightarrow y$$

$$y = x * h$$

$$Y(\omega) = X(\omega)\, H(\omega).$$

Parsevals thm :

$$\sum_n x(n)\, y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega)\, Y(\omega)\, d\omega$$

special case $x(n) = y(n)$.

$$\sum_n |X(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |X(\omega)|^2\, d\omega$$

<u>Ex</u> $\delta(n) \longrightarrow 1$

$x(n) = \left(\frac{1}{2}\right)^n u(n) \longrightarrow \dfrac{1}{1 - \frac{1}{2} e^{-j\omega}}$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = X(\omega)$$

$$1 + x + x^2 + \cdots + = \frac{1}{1-x}$$

$|x| < 1$

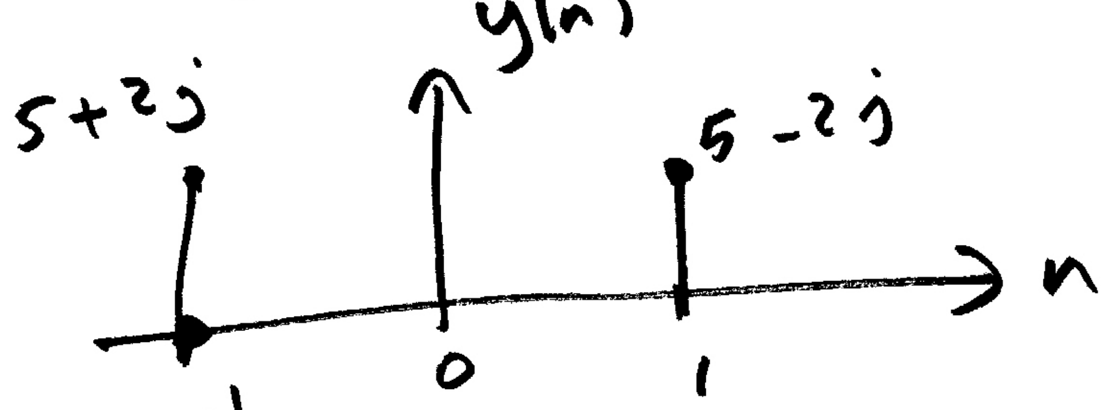
__Ex__ : $2^n u(n) = x(n)$

$\sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$ might or might __not__ converge.

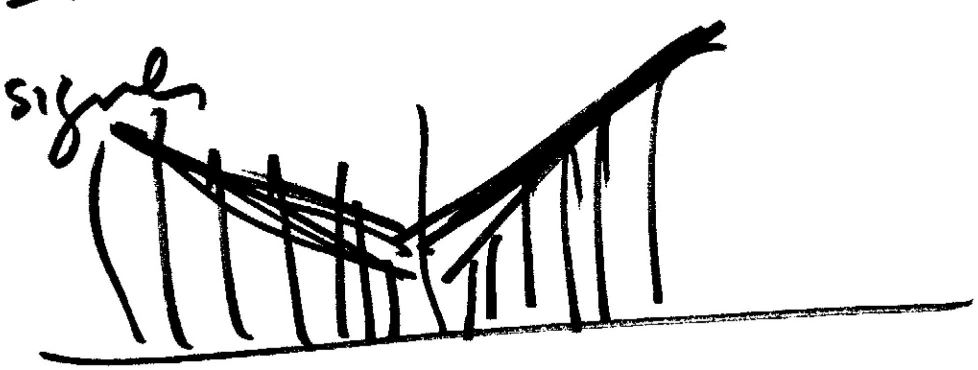
$\hookrightarrow$ DTFT for $2^n u(n)$ doesn't exist.

Symmetry Properties

Conjugate symmetric $\quad y(n) = y^*(-n)$

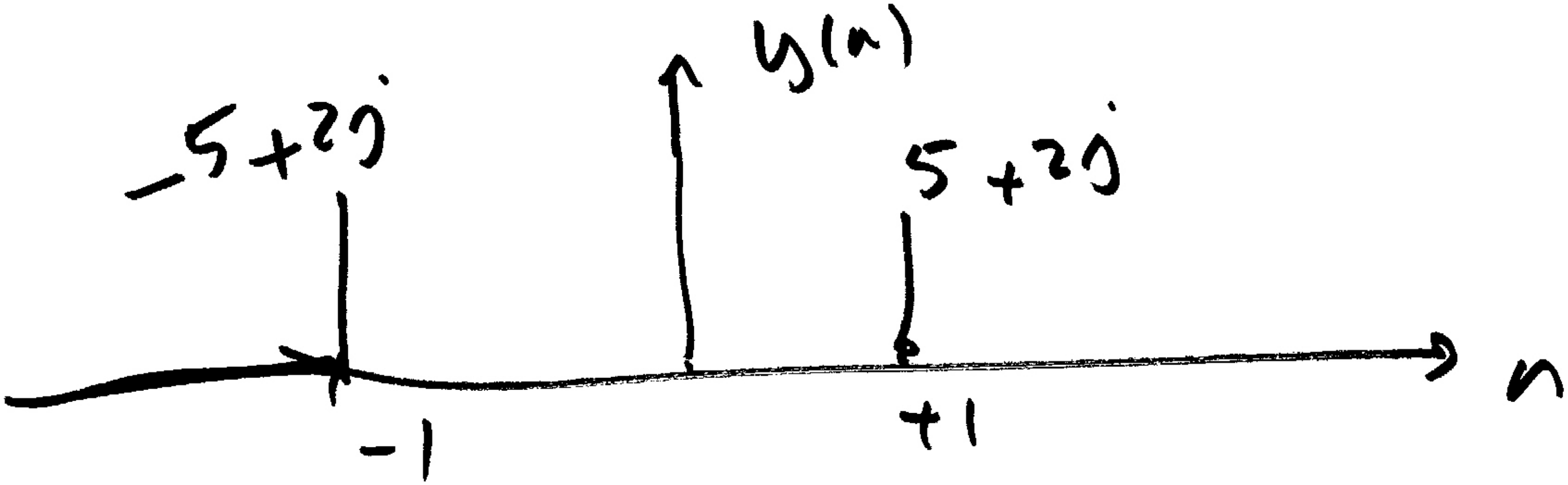


for real signals



Conjugate anti symmetric.

$$y(n) = -y^*(-n)$$



Any signal $x(n)$ can be decoupled into sum of conjugate symmetric & conjugate antisymetric part.

$Xe(n) \triangleq$ conjugate symmetric part of $x(n)$

$= \frac{1}{2}\left[x(n) + x^{*}(-n) \right] \rightarrow CS$

$= \frac{1}{2} \quad x^{*}(-n) + x(+n)$

$x_o(n) \triangleq$ Conjugate antisymmetric part of $x(n)$

$$= \frac{1}{2}\left[x(n) - x^*(-n) \right]$$

$$\cancel{\tfrac{1}{2}\left[-x^*(-n)\right.} + x(+n) \Big]$$

$$\boxed{x_e(n) + x_o(n) = x(n)}$$

$x_e(n) \to$ CS

$x_o(n) \to$ CAS

$$X(\omega) = X_e(\omega) + X_o(\omega)$$

$$X_e(\omega) \triangleq CS = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right]$$

$$X_o(\omega) \triangleq CAS = \frac{1}{2}\left[X(\omega) - X^*(-\omega)\right]$$

Symmetry Properties

$$x(n) \xleftrightarrow{\text{DTFT}} X(\omega)$$

$$\mathrm{Re}\{x(n)\} \xleftrightarrow{\text{DTFT}} X_e(\omega)$$

$$j\,\mathrm{Im}\{x(n)\} \xleftrightarrow{\text{DTFT}} X_o(\omega)$$

$$x_e(n) \xleftrightarrow{\text{DTFT}} \mathrm{Re}\{X(\omega)\}$$

$$x_o(n) \xleftrightarrow{\text{DTFT}} j\,\mathrm{Im}\{X(\omega)\}$$

---

if $x(n)$ is real $\Longrightarrow$ $x(n) = \mathrm{Re}\{x(n)\}$

$$X(\omega) = X_e^*(\omega) = \frac{1}{2}\left[X(\omega) + X^*(-\omega)\right]$$

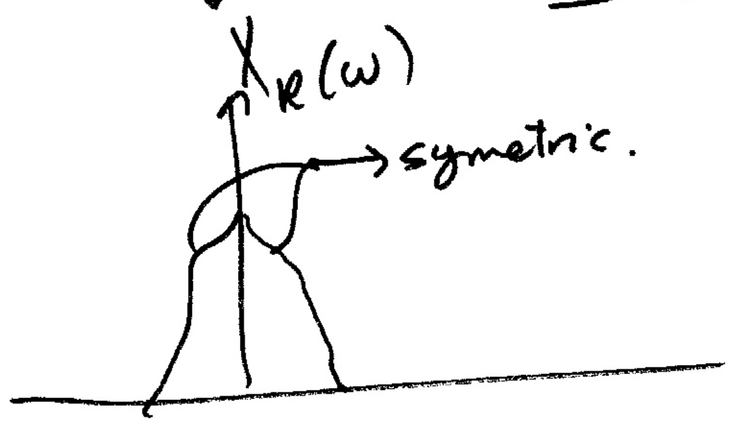
$$\Longrightarrow X(\omega) = X^*(-\omega)$$

If $x(n)$ real $\Longrightarrow$ F.T. is Conjugate Symetric.

$$Re\{X(\omega)\} = Re\{X^*(-\omega)\}$$

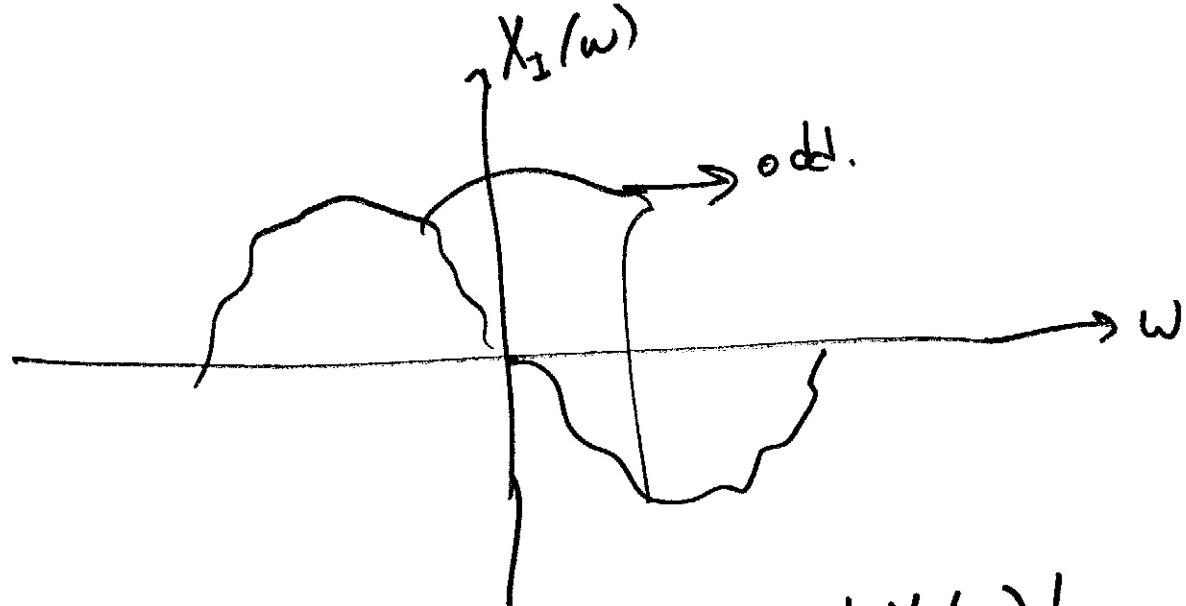
$$\Longrightarrow (1)\ \underset{\text{real part}}{X_R}(\omega) = X_R(-\omega)$$

$\Longrightarrow$ Real part of $X(\omega)$ is even.



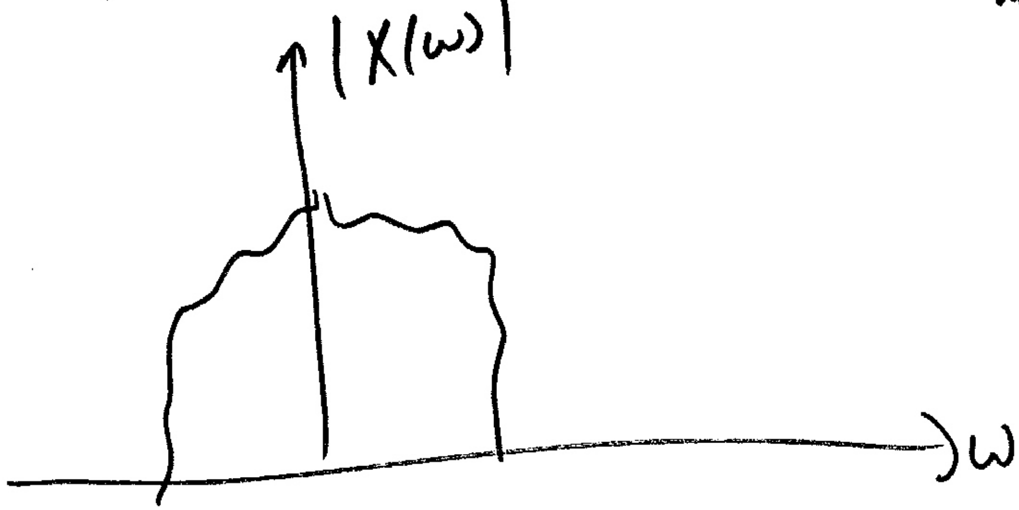
$$X(\omega) = X^*(-\omega) \Longrightarrow (2)\ \underset{\text{imaginary}}{X_I}(\omega) = -X_I(-\omega)$$

Imaginary part of $X(\omega)$ is odd

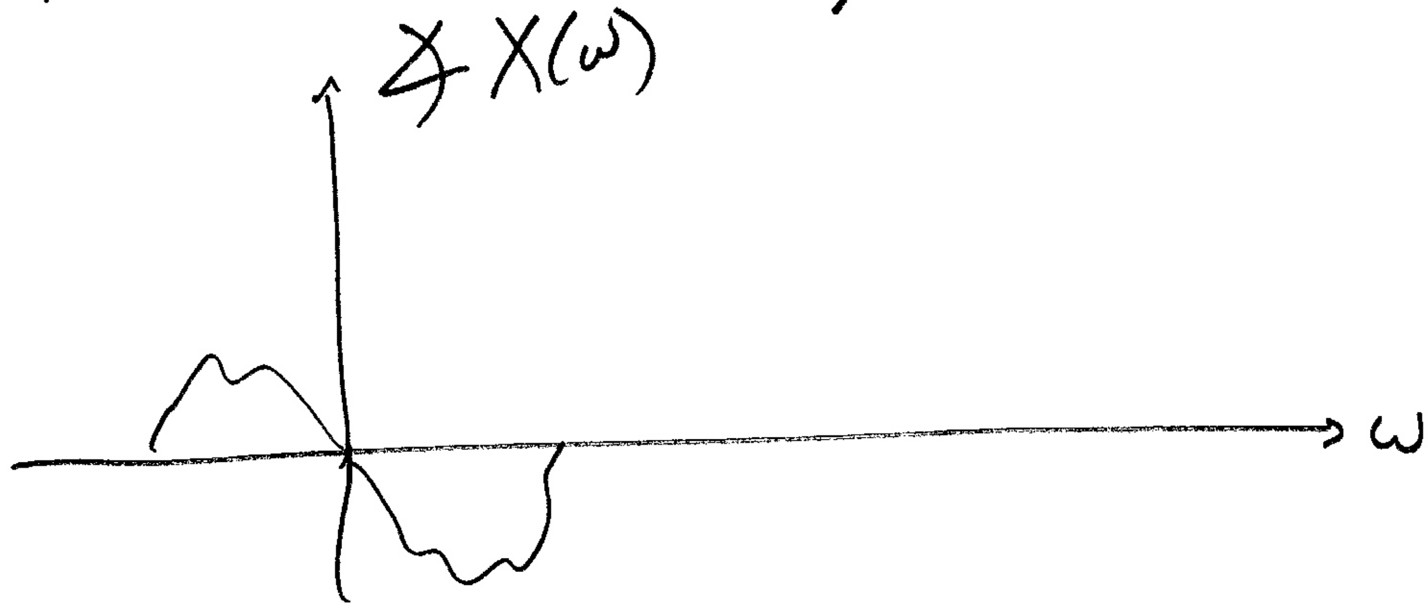


$X(\omega) = X^*(-\omega) \Longrightarrow$ ③ $|X(\omega)| = |X(-\omega)|$

magnitude of F.T. is even. for $x(n)$ real



$$X(\omega) = X^*(-\omega) \Longrightarrow \text{(4)} \quad \measuredangle X(\omega) = -\measuredangle X(-\omega)$$



If $x(n)$ real $\Longrightarrow$

1. $X_R(\omega)$ is symetric
2. $X_I(\omega)$ is anti symetric
3. $|X(\omega)|$ is symetric
4. $\measuredangle X(\omega)$ is anti symetric

Z. Transform

Motivation: F.T. doesn't exist for some signal.

$x(n) = 2^n u(n)$.

or

$x(n) = a^n u(n)$ $\quad |a| > 1$

F.T. doesn't exist.

$$x(n) \longleftrightarrow \mathcal{Z}\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

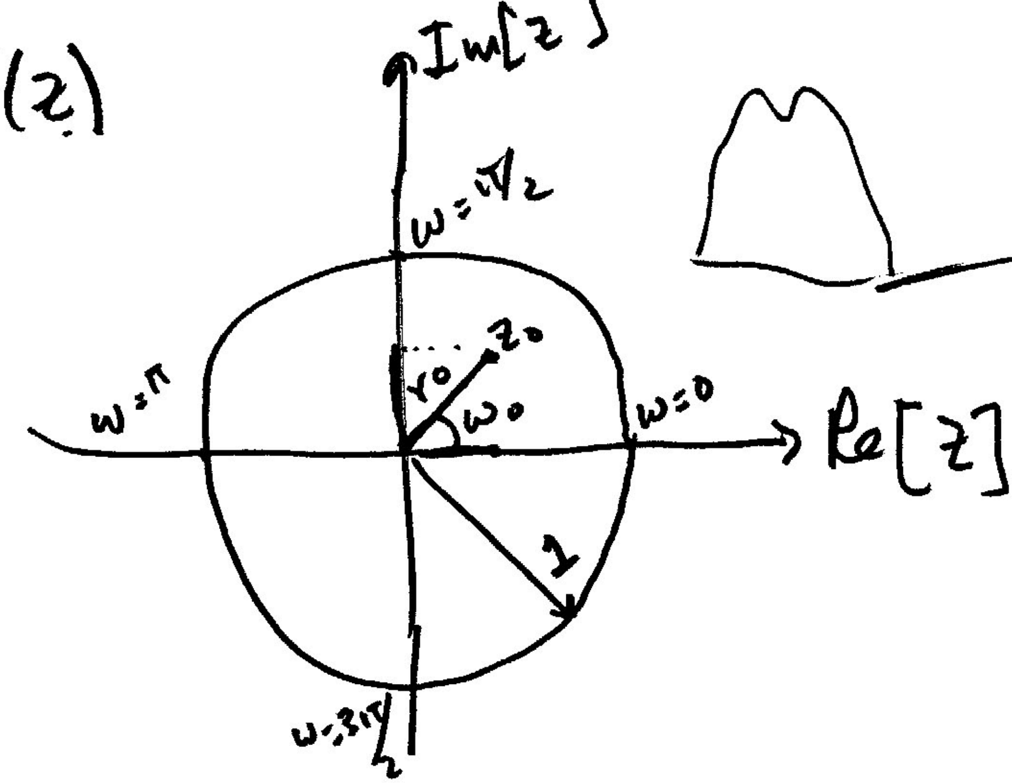
complex variable.

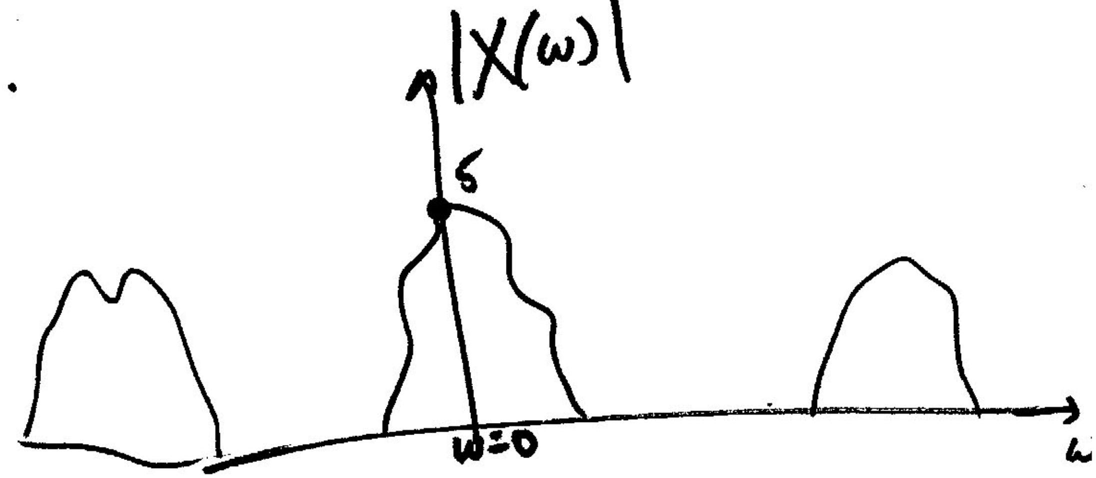
$$X(\omega) = \text{D.T.F.T.}\{x(n)\} = \sum_n x(n) e^{-j\omega n}$$

$$\left[X(z)\right]_{z=e^{-j\omega}} = X(\omega) \longrightarrow \text{Real variable.}$$

Z.T. evaluated at unit circle corresponds To D.T.F.T.

$X(z)$





z = complex variable

$= Re + j\, Im.$

$z = re^{j\omega}$

$z_0 = r_0 e^{j\omega_0}$

$\Rightarrow |z_0| = r_0 \quad \measuredangle z_0 = \omega_0$

Does $X(z)$ converge? For what values of z does $X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$ converge?

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) \left(re^{+j\omega}\right)^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(x(n)\, r^{-n} \right)}_{\text{D.T.F.T of } x(n) r^{-n}} e^{-j\omega n}$$

Remember: D.T.F.T. of $x(n)\, r^{-n}$ exists (converges)

if $x(n)\, r^{-n}$ is abs. summable.

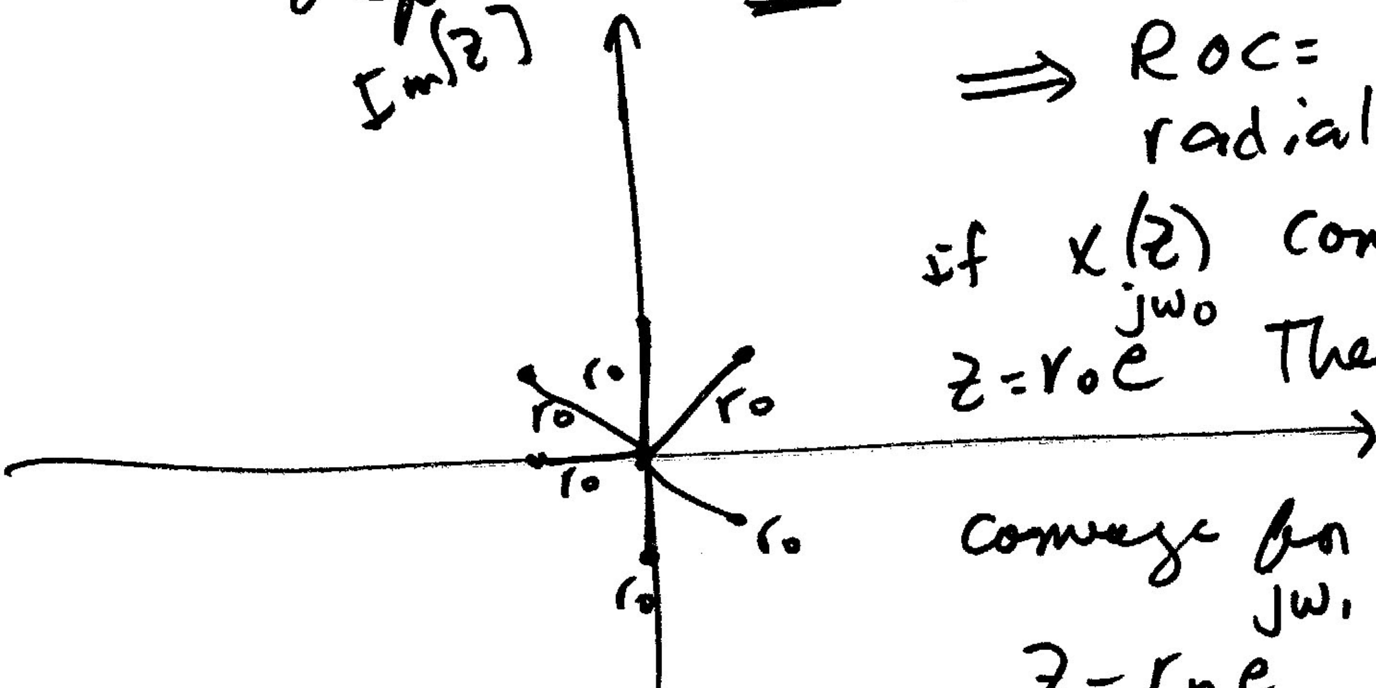
i.e. $$\sum_n \left| x(n)\, r^{-n} \right| < \infty$$

If $\sum_n \left| x(n)\, r^{-n} \right| < \infty \Longrightarrow$ Z.T. exists & converges.

$\Longrightarrow$ Conclusion: Convergence only depends on $\underline{r}$ & NOT ω.

$\Longrightarrow$ ROC = always radially symmetric

if $X(z)$ converges for $z = r_0 e^{j\omega_0}$ Then it also converges for $z = r_0 e^{j\omega_1}$, $r_0 e^{j\omega_2}$



$\Longrightarrow$ NOT angle dependent.

ROC = Region of convergence.

Region in z domain for which

$\sum_n x(n) z^{-n}$ converges.

$\Rightarrow$ ROC possibilities:

- inside of some circle
- outside of some circle.
- Between 2 circles $\rightarrow$ ring.
- A point i.e. origin

Region of Convergence for Z.T.

<u>Claim</u> $X(z)$. by itself doesnot specify uniquely a sequence..

$$X(z) + \underline{ROC} \longleftrightarrow x(n)$$

$$X(z) + ROC1 \longleftrightarrow x_1(n)$$

$$X(z) + ROC2 \longleftrightarrow x_2(n)$$

<u>Example</u>: $X(z) = \dfrac{1}{1 - az^{-1}}$

Region of Convergence for Z.T.

<u>Claim</u> $X(z)$. by itself doesnot specify uniquely a sequence..

$$X(z) + \underline{ROC} \longleftrightarrow x(n)$$

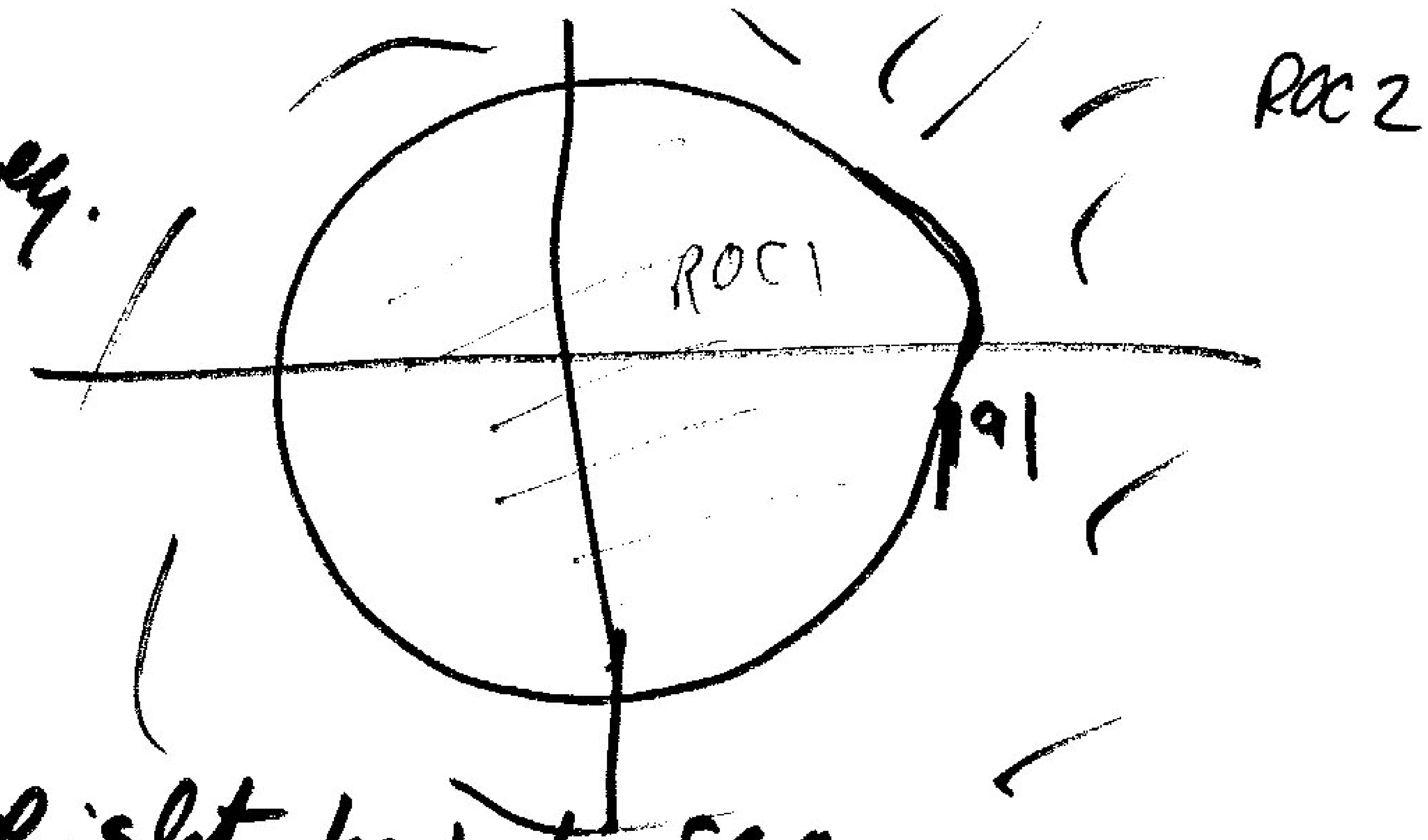
$$X(z) + ROC1 \longleftrightarrow x_1(n)$$

$$X(z) + ROC2 \longleftrightarrow x_2(n)$$

<u>Example</u>: $X(z) = \dfrac{1}{1 - az^{-1}}$

ROC1 : ~~Right~~ left handed seq.

ROC2 : ~~Left~~ Right handed seq.

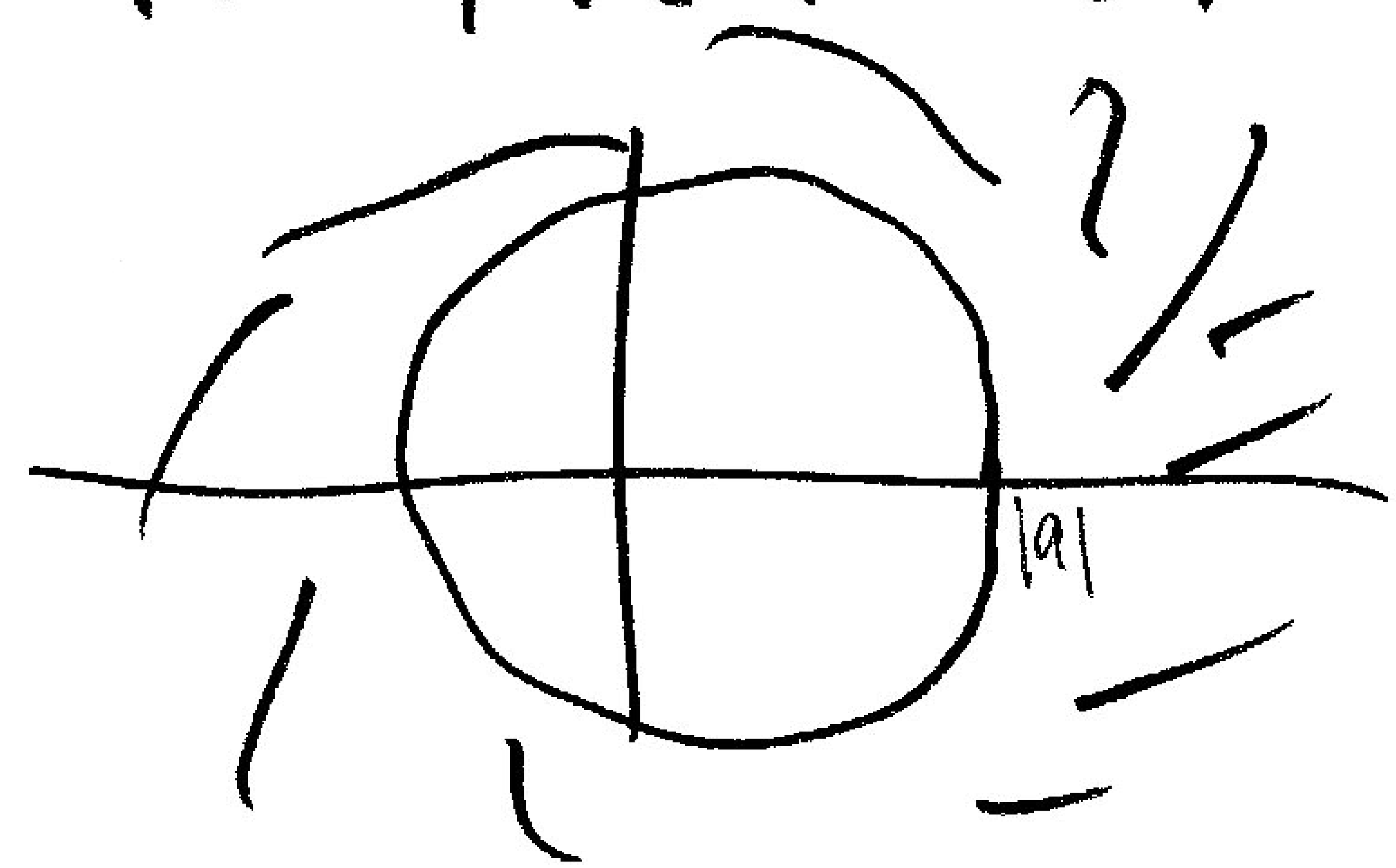


$x(n) = a^n u(n)$ ← Right handed seq.

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1-a z^{-1}}$$

converges if $|a z^{-1}| < 1 \Rightarrow |z| > |a|$

→ ROC2



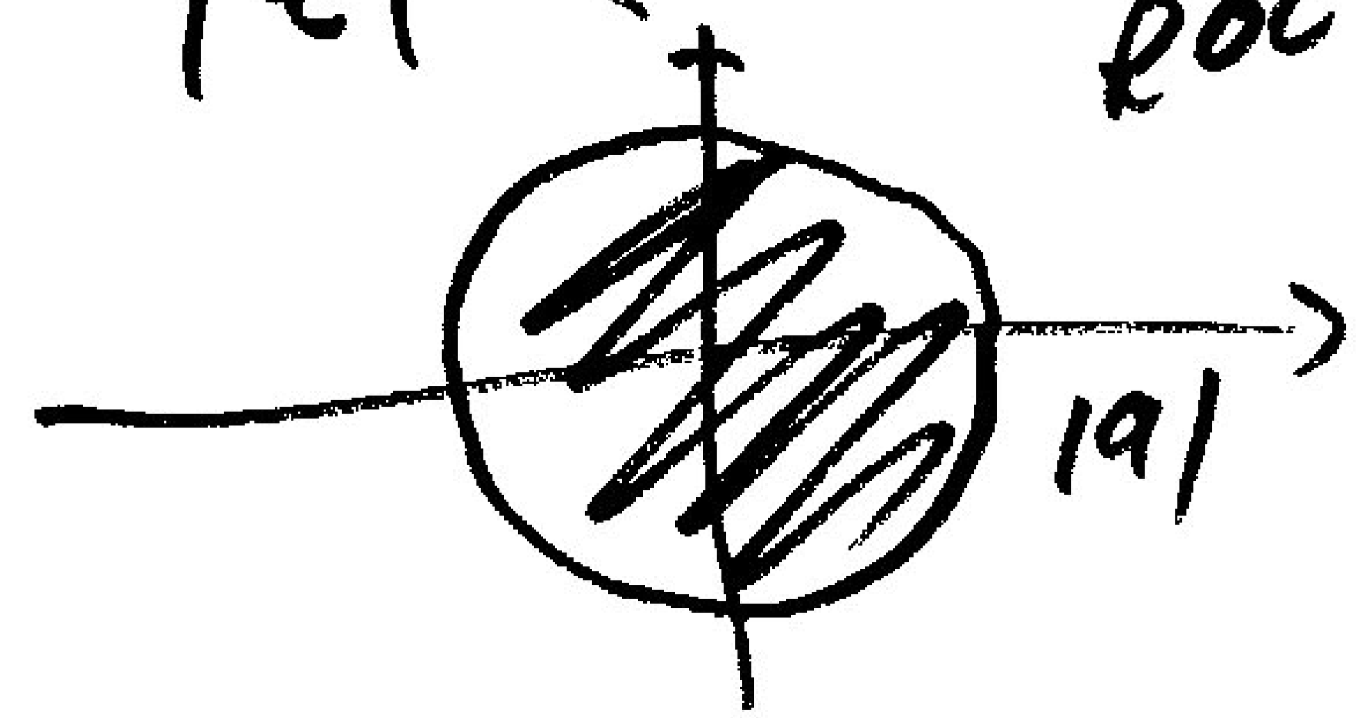
Ex $x(n) = -a^n u(-n-1)$ ← lefthanded

$$X(z) = \sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \underbrace{\sum_{m=0}^{\infty} (a^{-1}z)^m}$$

chge of variable $m=-n$

Convergence $|a^{-1}z| < 1 \Longrightarrow |z| < |a|$ ROC 1

$$X(z) = \frac{1}{1-az^{-1}}$$



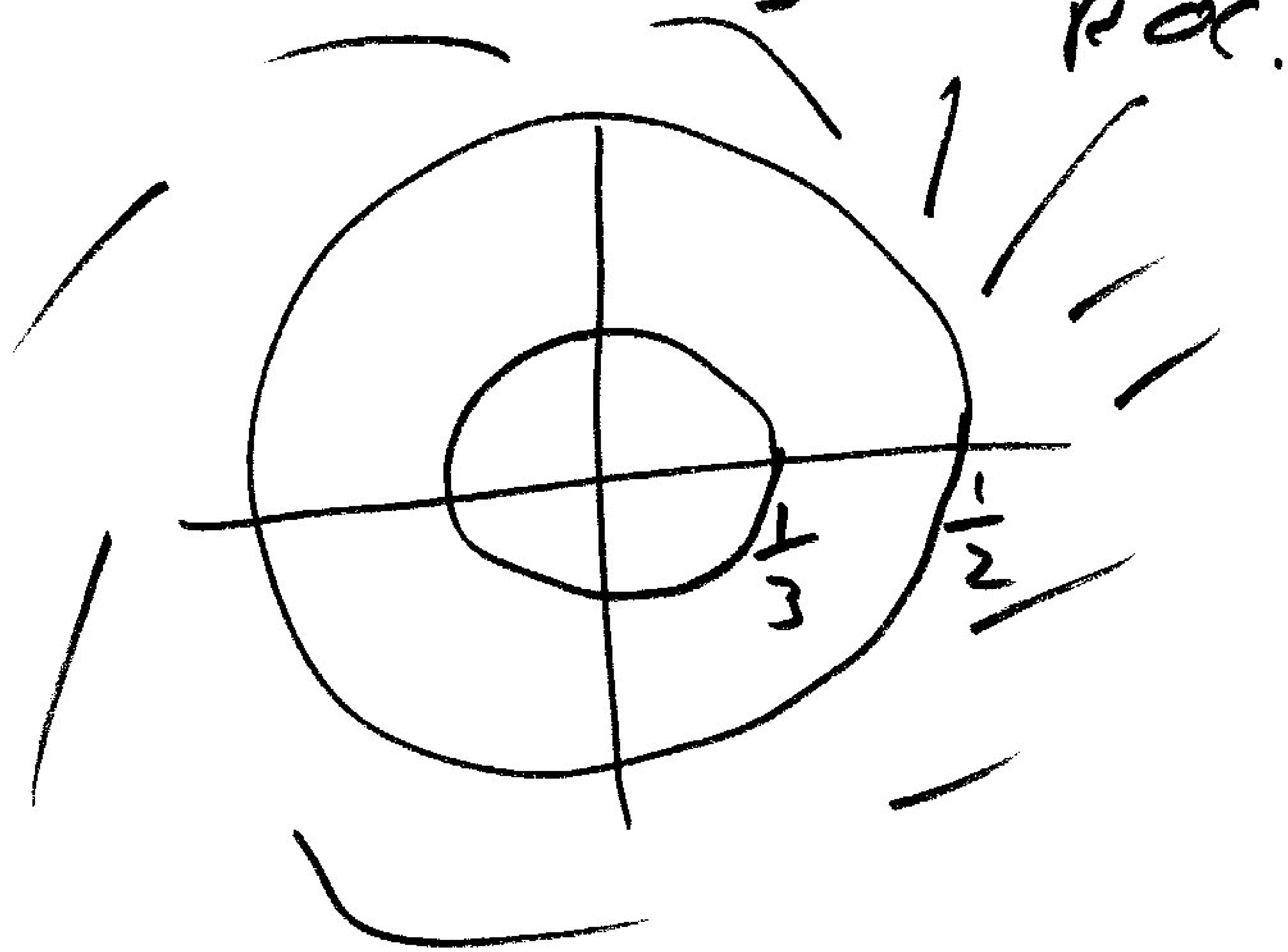
Ex

$x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

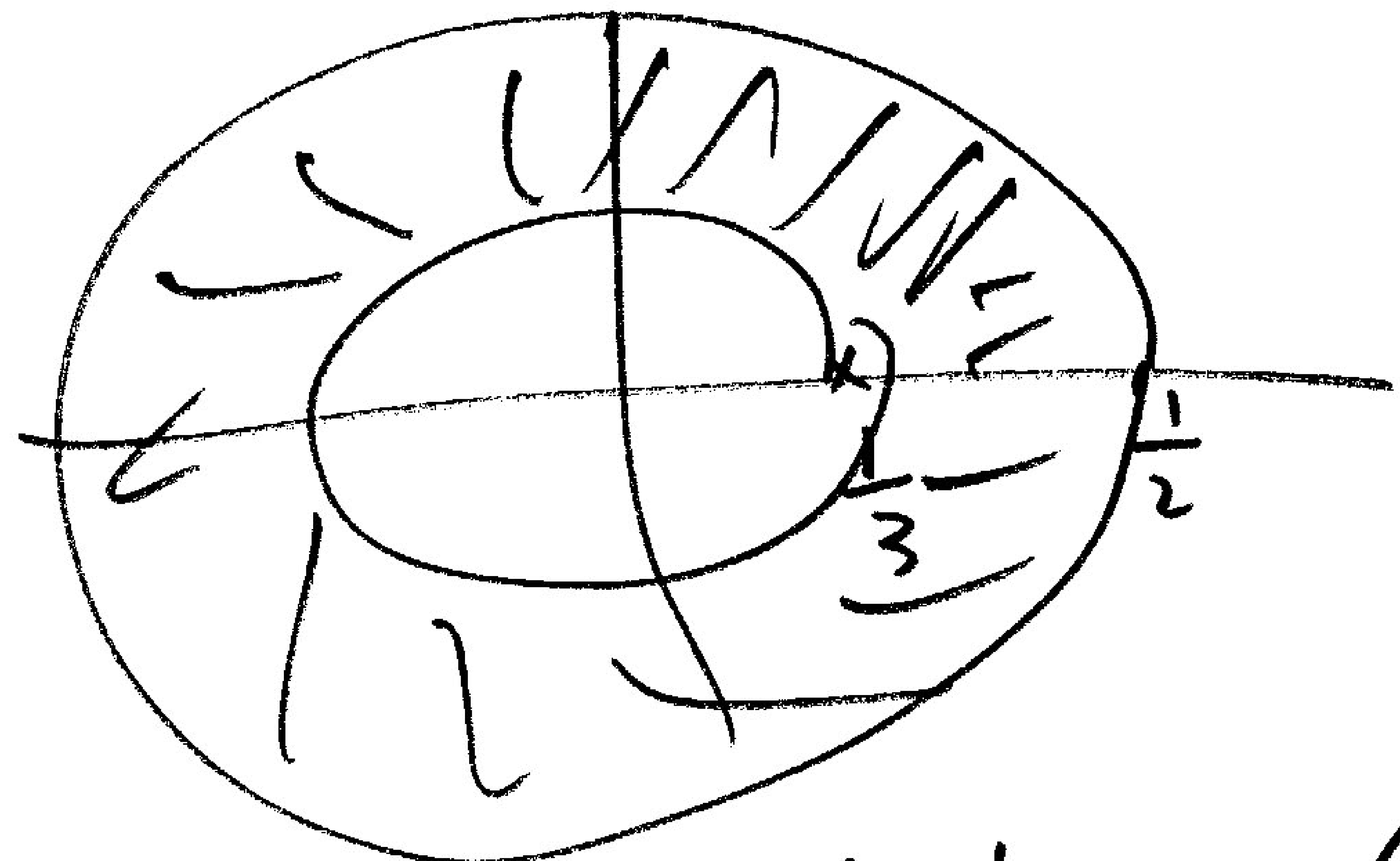
$\downarrow$ converges $|z| > \frac{1}{2}$

$\downarrow$ converges $|z| > \frac{1}{3}$



ROC: $|z| > \frac{1}{2}$

Ex $x(n) = \underbrace{\left(-\frac{1}{3}\right)^n u(n)}_{|z| > \frac{1}{3}} - \underbrace{\left(\frac{1}{2}\right)^n u(-n-1)}_{|z| < \frac{1}{2}}$

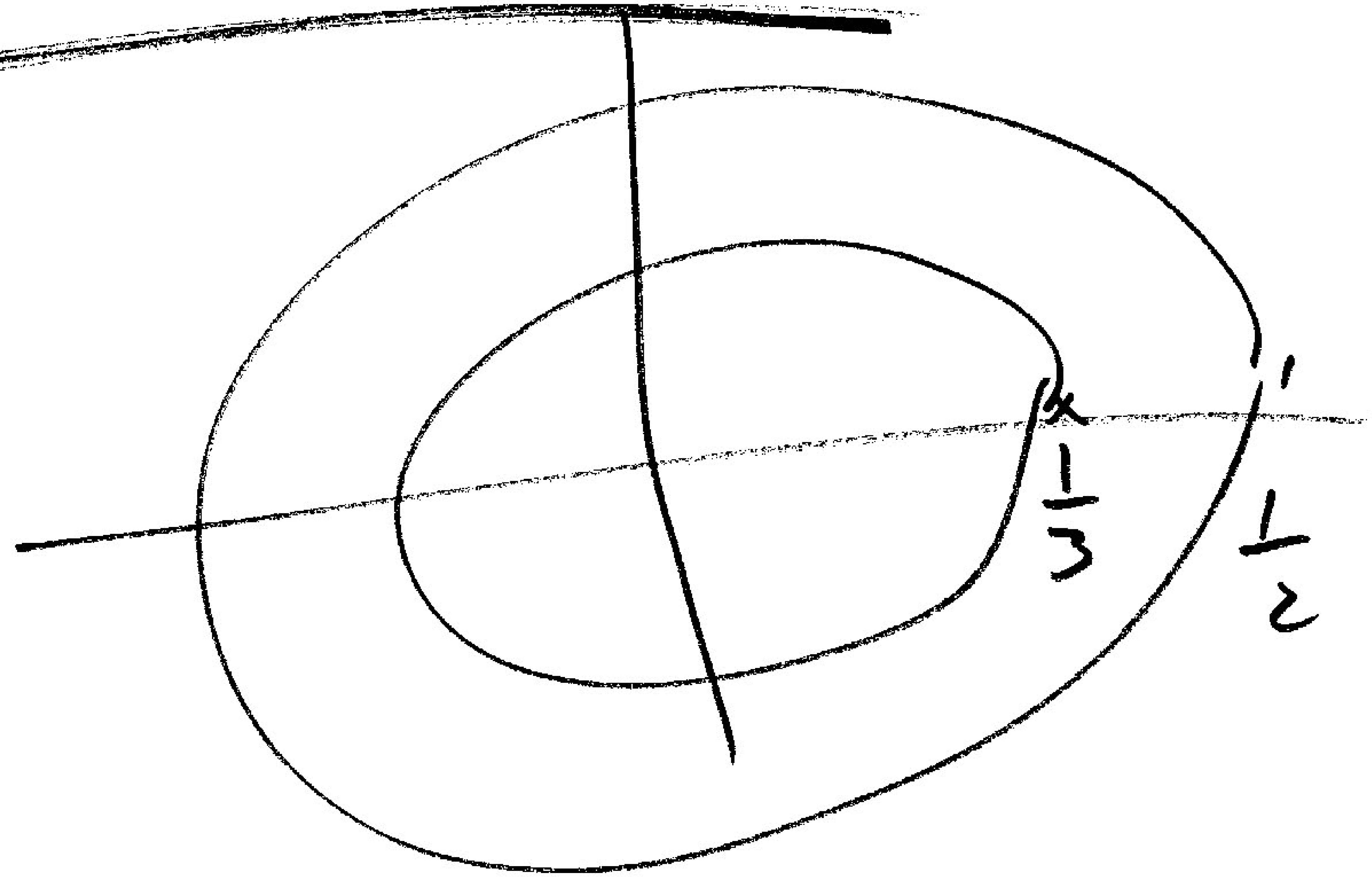


ROC is a ring.

Both handed Seq.

Ex $x(n) = \underbrace{\left(-\frac{1}{2}\right)^n u(n)}_{|z| > 1/2} - \underbrace{\left(\frac{1}{3}\right)^n u(-n-1)}_{|z| < \frac{1}{3}}$

Z.T. doesnot Exist.



observations

RHS $\Longrightarrow$ ROC outside of some circle.

LHS $\Longrightarrow$ ROC is inside of some circle.

BHS $\Longrightarrow$ Ring or doesn't exist.

Ex $x(n)$: Finite length seq.

$$X(z) = \sum_{n=M}^{N} x(n) z^{-n} = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + \cdots + x(-1) z + x(-2) z^{2} + \cdots$$

Converges everywhere except maybe at $z = 0$, $z \to \infty$

Ex

$$x(n) = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$X(z) = \frac{1-(az^{-1})^N}{1-az^{-1}}$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-N+1}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots + \frac{a^{N-1}}{z^{N-1}}$$

ROC: everywhere except for $z=0$.

Ex

$$x(n) = \begin{cases} a^n & -N \le n \le 0 \\ 0 & \text{elsewhere.} \end{cases}$$

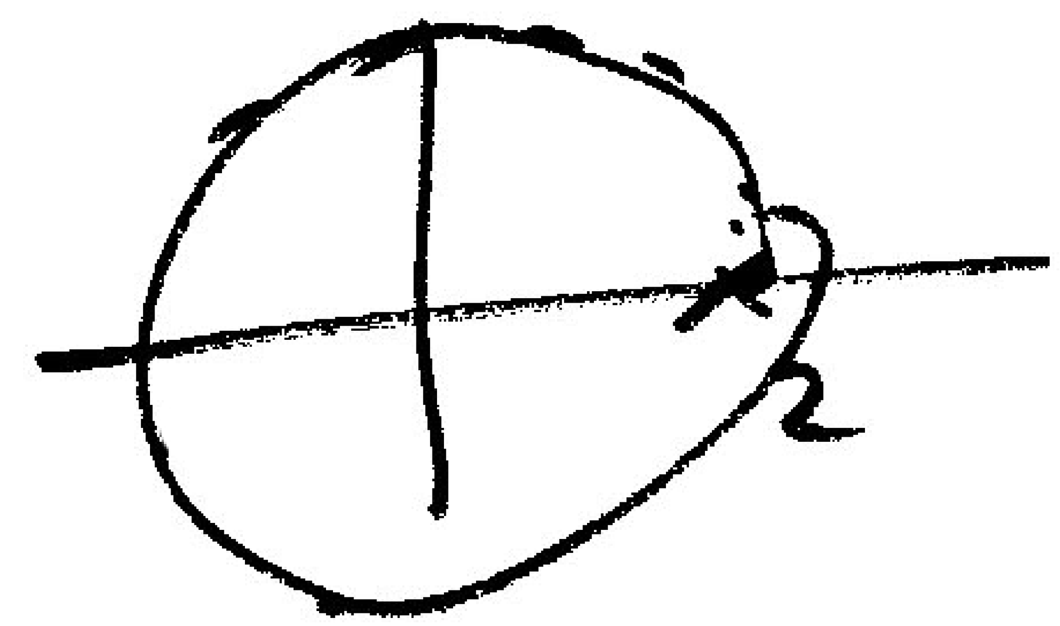
$$X(z) = \sum_{n=0}^{-N} a^n z^{-n} = 1 + a^{-1} z + a^{-2} z^2 + \cdots$$

ROC: converges everywhere except at $z \to \infty$

Properties of ROC :

1. ROC is a fn of r and not ω
2. F.T. exists if ROC of the Z.T. includes unit circle
3. ROC cannot include a pole

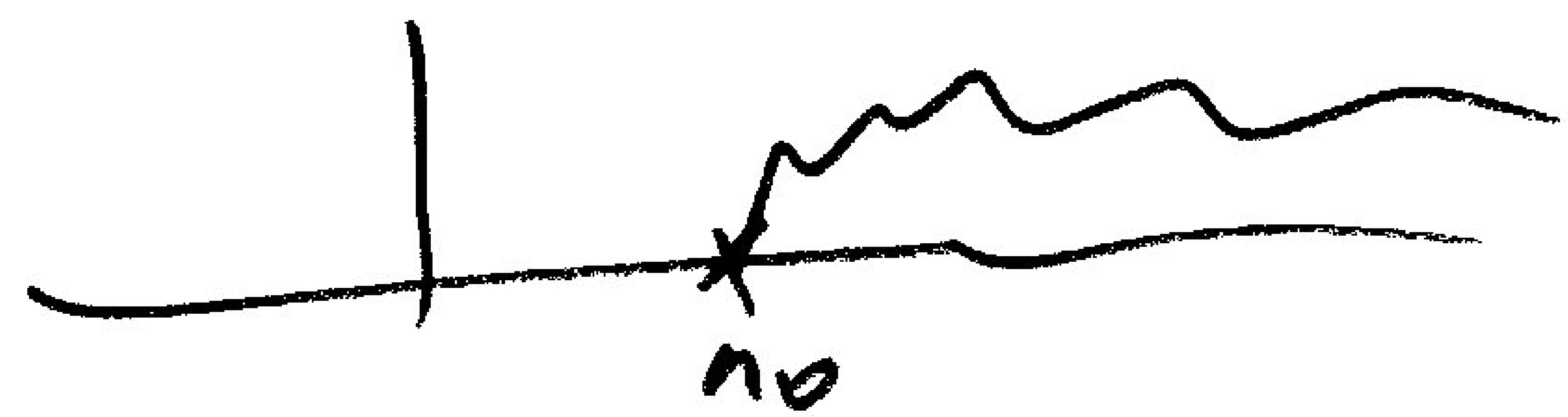
$$\frac{1}{1-\frac{1}{2}z}$$



4. Finite length seq
   ROC: everywhere except at $z=0$ or $z\to\infty$.

<u>Show</u>: If seq. <u>RHS</u> $\Longrightarrow$ ROC <u>outside of some circle</u>.

- $x(n)$ RHS : non zeros for some n_0 and larger indices than n_0



- ROC outside of some circle: $\Longleftrightarrow$
  If $X(z)$ converges for r_0, it also converges for $\forall\, r > r_0$

Assume $X(z)$ converges for some r_0,
show that it also converges for $r > r_0$

$$\sum_{n=-\infty}^{+\infty} \left| x(n) r_0^{-n} \right| < \infty \qquad \text{True}$$

Invoke the RHS assumption

$$\sum_{n=n_0}^{\infty} \left| x(n) r_0^{-n} \right| < \infty \qquad \text{True.}$$

<u>Show</u>: $\boxed{r > r_0}$ $\quad \sum_{n=n_0}^{\infty} \left| x(n) r^{-n} \right| < \infty$

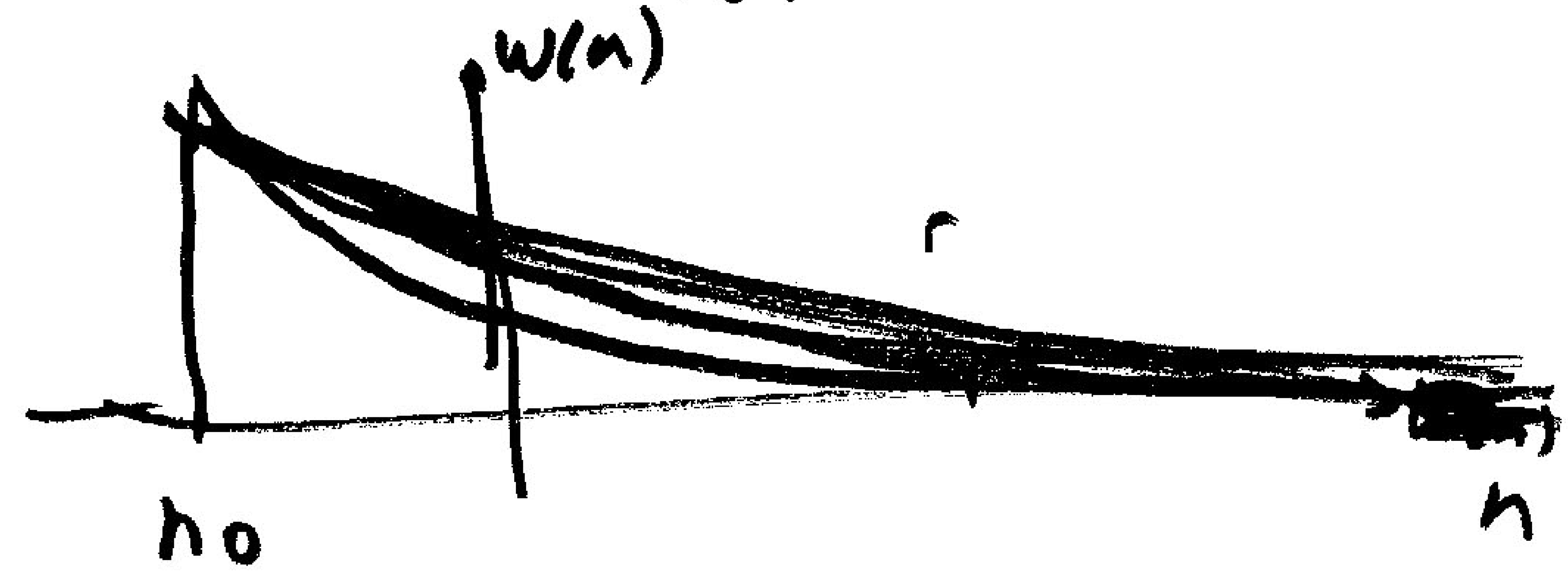
$$\left| x(n) r^{-n} \right| = \left| x(n) r_0^{-n} \left(\frac{r}{r_0}\right)^{-n} \right|$$

$$= \left| x(n) r_0^{-n} \right| \left| \left(\frac{r}{r_0}\right)^{-n} \right|$$

Show $\boxed{r > r_0}$ $\sum_{n=n_0}^{\infty} \left|x(n) r_0^{-n}\right| \underbrace{\left|\left(\frac{r}{r_0}\right)^{-n}\right|}_{}^{w(n)} < \infty$

$r > r_0 \Rightarrow \left|\frac{r_0}{r}\right| < 1 \Rightarrow \left|\frac{r}{r_0}\right|^{-1} < 1$

<u>RHS</u>



$\Longrightarrow$ QED