Optimal FIR Filter Design

Type I Generalized Linear Phase Filter.

LPF

$$H(\omega) = H_m(\omega)\, e^{j\beta}\, e^{-j\alpha\omega}$$

Assume Type I $\Longrightarrow \beta = 0$,

$\boxed{\begin{aligned} N &= \#\text{ of Taps is odd.} \\ &= 2M+1 \end{aligned}}$

$$h(n) = h(N-n-1)$$

$$H_m(\omega) = \boxed{\sum_{n=0}^{M} a(n) \cos(\omega n) \triangleq G(\omega)}$$

$a(0) = h(0)$

$a(n) = 2h(M-n)$ all other n.

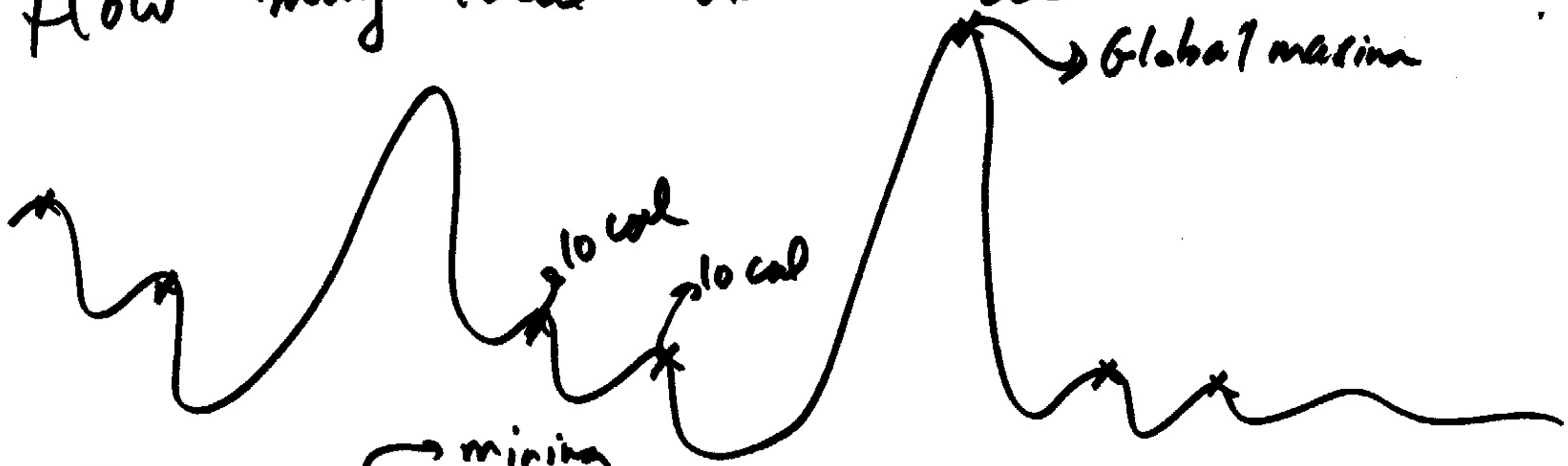
$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$H(\omega) = G(\omega)\, e^{-j\alpha\omega}$$

Observations on $G(w)$

① $G(w)$ is a continous fn of w and is as many times differentiable as you want.

② How may local extrema does $G(w)$ have?



Extrem → minima
→ maxima.

Express $\cos(wn)$ as sum of powers of $\cos w$.

$$\cos(2w) = 2\cos^2 w - 1$$

$$\cos(3w) = \cos(2w + w) = \cos 2w \cos w - \sin 2w \sin w$$

$$= \cos w\left[2\cos^2 w - 1\right] - 2\sin^2 w \cos w$$

$$= 2\cos^3 w - \cos w - 2\cos w\left[1 - \cos^2 w\right]$$

$$= 4\cos^3 w - 3\cos w$$

Generally $\cos(wn)$ as sum of powers of $\cos w$

$$\cos(wn) = \sum_{i=0}^{n} \eta_i (\cos w)^i$$

Tchebychev
"Chebyshev"

$$G(\omega) = \sum_{n=0}^{M} a(n) \left[\sum_{i=0}^{n} \eta_i \; (\cos\omega)^i \right]$$

$$G(\omega) = \sum_{n=0}^{M} \gamma(n) \; (\cos\omega)^n$$

γ depends on η and a

To compute local extrema of $G(\omega)$ take the derivative and set to zero.

$$\frac{dG(\omega)}{d\omega} = 0 \Rightarrow \sum_{n=0}^{M} \gamma(n) \, n \, (\cos\omega)^{n-1} (-\sin\omega)$$

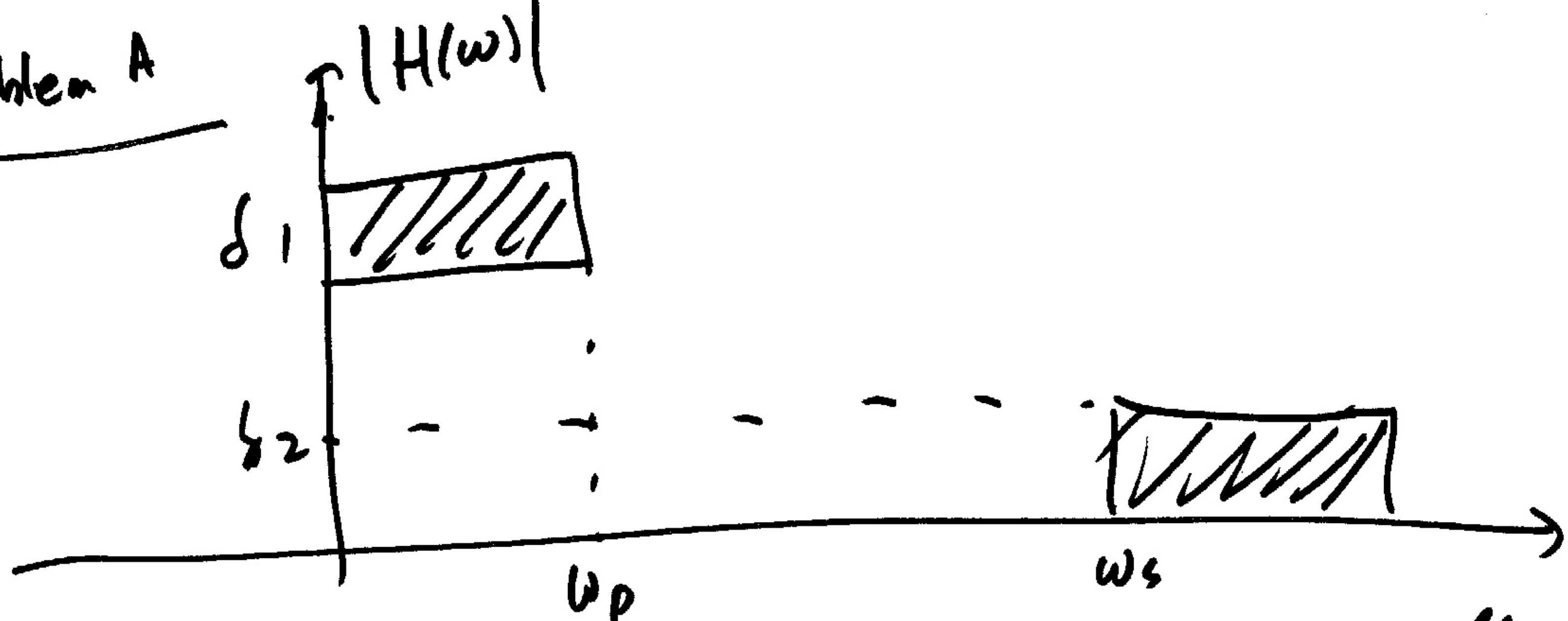
$$= 0 \Longrightarrow$$

$$\begin{cases} \sin\omega = 0 \implies \omega = 0, \pi \\ \underbrace{\sum_{n=0}^{M} \gamma(n)\, n \, (\cos\omega)^{n-1}}_{\text{Polynomial in } \cos\omega = x} = 0 \longrightarrow \text{Max of } M-1 \text{ zeros (real).} \end{cases}$$

$\Rightarrow$ $\boxed{\text{total \# of local extrema for } G(\omega) \text{ is } (M-1) + 2 = M+1}$

Problem Statement Optimal FIR Filter Design

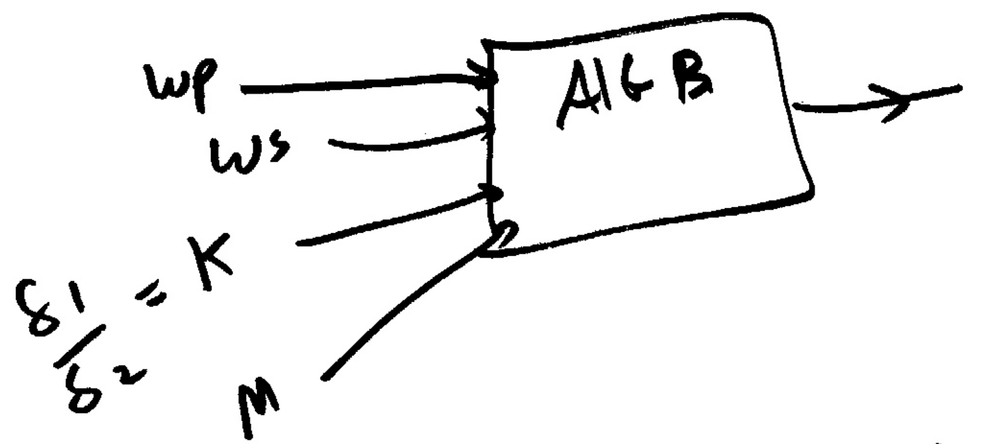
Problem A



Given ω_s, ω_p, δ_1, δ_2, Determine coeff of $G(\omega)$ i.e. $a(n)$ such that M is minimized

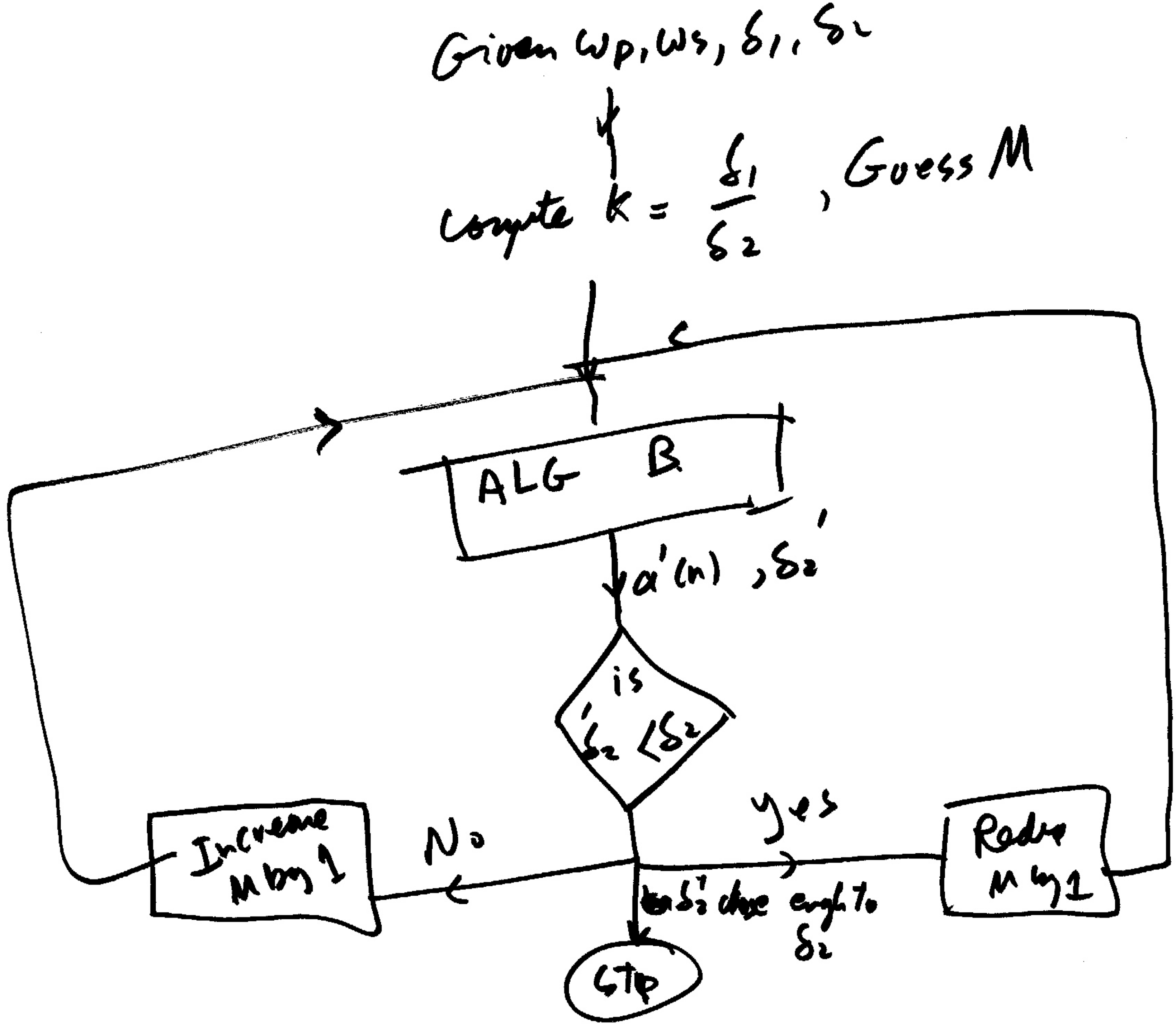
<u>Problem B</u> Given w_p, w_s, M, $K = \frac{\delta_1}{\delta_2}$

Determin $a(n)$ such that δ_2 is minimized.



box

Show that if I had a box that solves Problem B, Then I can use it To solve Problem A.



Given $\omega_p, \omega_s, \delta_1, \delta_2$
Compute $k = \frac{\delta_1}{\delta_2}$, Guess M
ALG B
$\alpha'(n)$, δ_2'
is $\delta_2' < \delta_2$
yes
No
Increase M by 1
Reduce M by 1
δ_2' close enough to δ_2
Stop

Problem C: $E(\omega) = W(\omega)\left[G(\omega) - D(\omega)\right]$

where $W(\omega)$ = positive weighting function $\begin{cases} \frac{1}{K} & I_1 \\ 1 & I_2 \end{cases}$

$$G(\omega) = \sum_{n=0}^{M} a(n)\cos(\omega n)$$

$D(\omega)$ = Desired Freq. Response = $\begin{cases} 1 & I_1 \\ 0 & I_2 \end{cases}$

$$\boxed{\text{Problem C: Find } a(n) \text{ To minimize } \max_{\omega \in F} |E(\omega)|}$$

when $F = I_1 \cup I_2$ is a subset of a closed Interval $0 \le \omega \le \pi$

I_1 = passband, I_2 = stopband.

Optimum FIR Filter Design

Problem C: $E(\omega) = W(\omega)\left[G(\omega) - D(\omega)\right]$

$$G(\omega) = \sum_{n=0}^{M} a(n) \cos(\omega n)$$

$$W(\omega) = \text{positive Weighting fn} = \begin{cases} \frac{1}{K} & I_1 \\ 1 & I_2 \end{cases}$$

$$D(\omega) = \text{Desired. Freq. Response} = \begin{cases} 1 & I_1 \\ 0 & I_2 \end{cases}$$

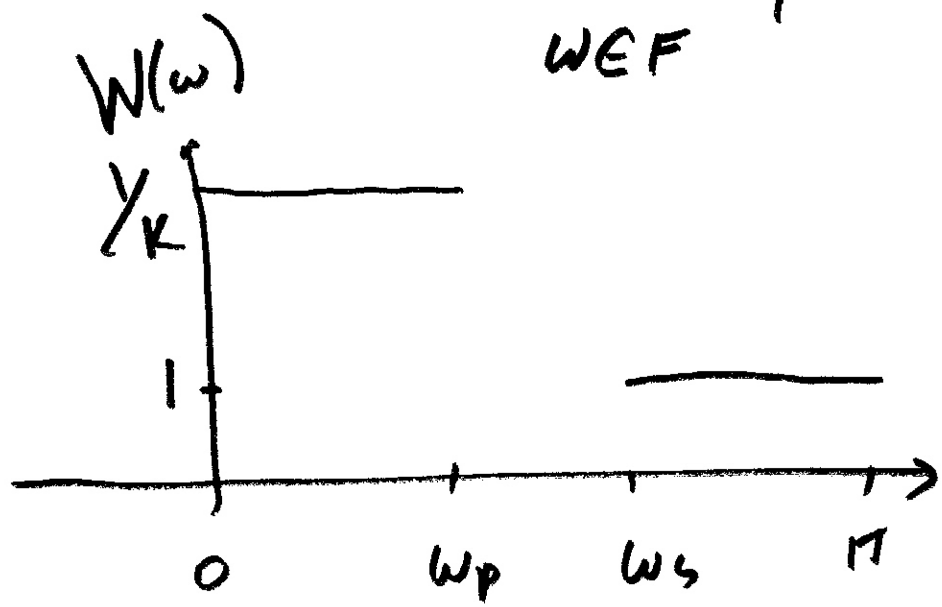
Find $a(n)$ To minimize. $\max_{\omega \in F} |E(\omega)| = \theta$

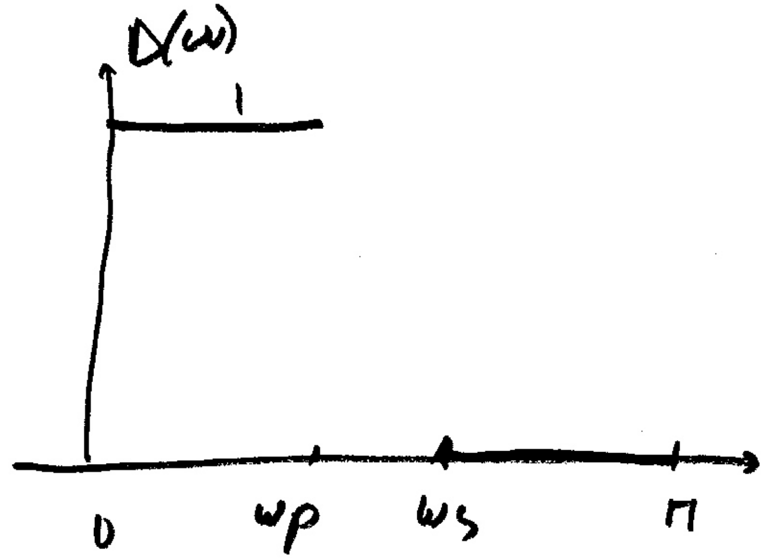
$F = I_1 \cup I_2$ = subset of closed interval $0 \le \omega \le \pi$

Show: Problem C is equivalent to Problem B.

Start to show $\Theta = \delta_2$

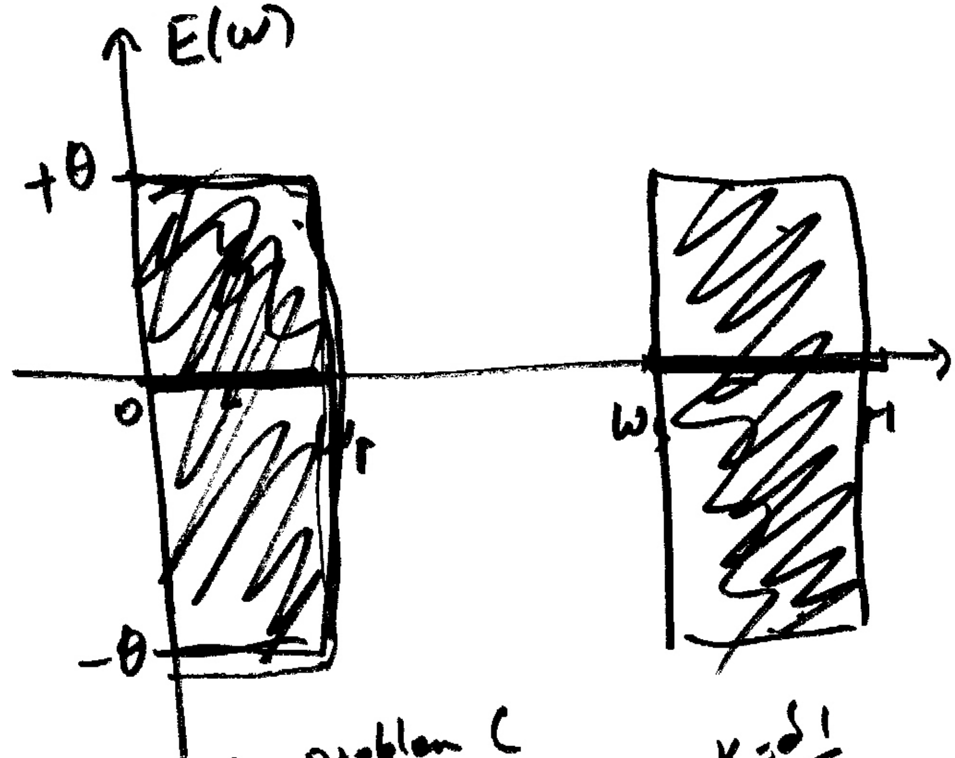
$$\max_{\omega \in F} |E(\omega)| \overset{??}{=} \delta_2$$

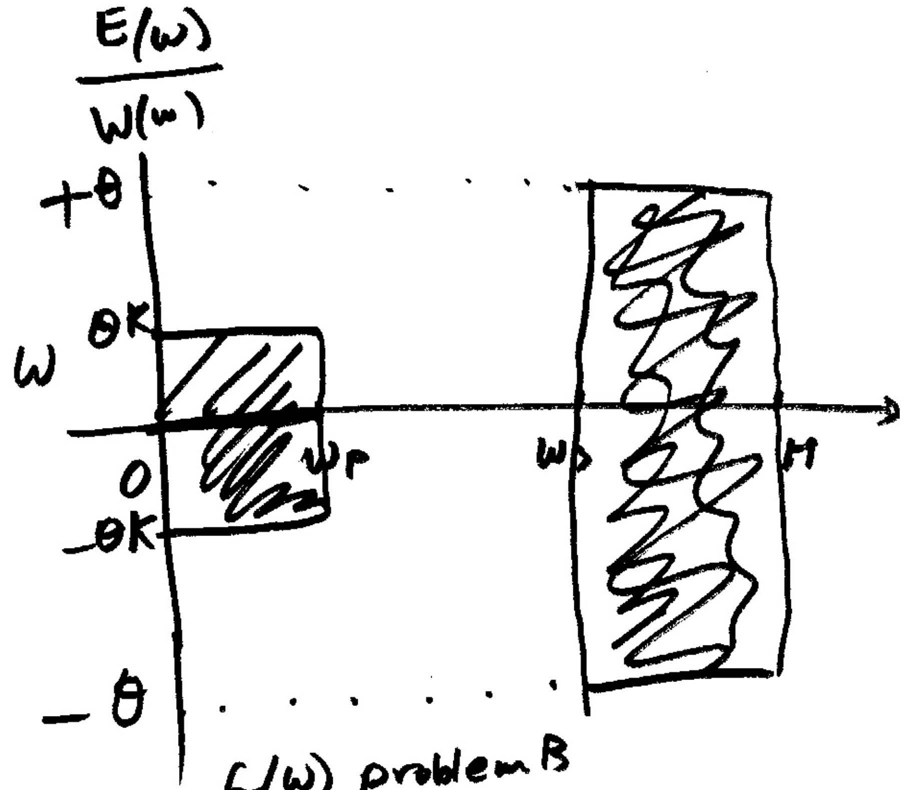




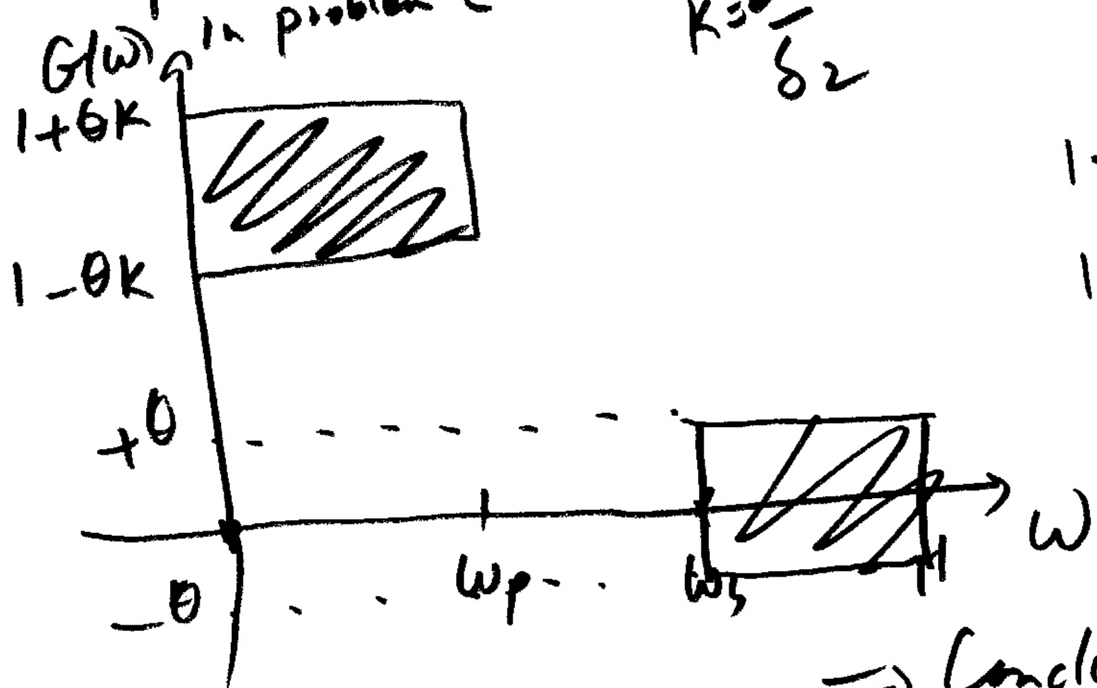
$$E(\omega) = W(\omega)\left[G(\omega) - D(\omega)\right] \Rightarrow \frac{E(\omega)}{W(\omega)} = G(\omega) - D(\omega)$$

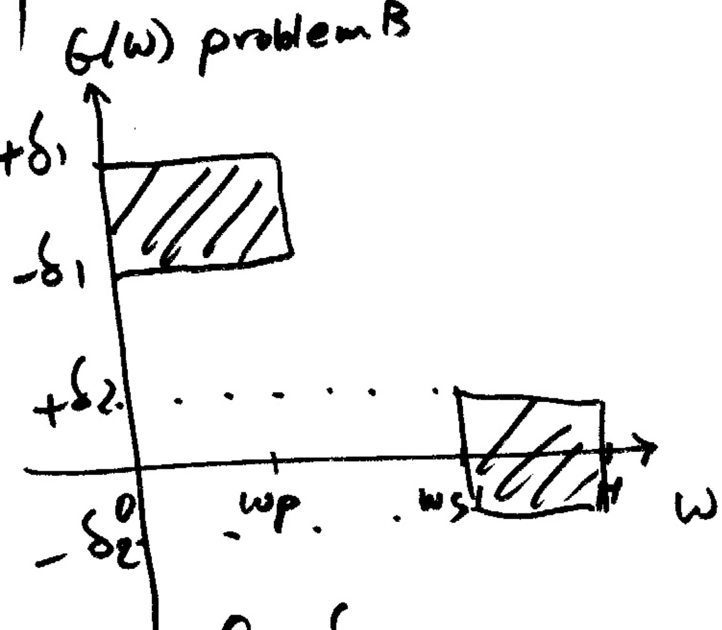
$$\Rightarrow G(\omega) = D(\omega) + \frac{E(\omega)}{W(\omega)}$$





$K = \frac{\delta_1}{\delta_2}$





$\Rightarrow$ Conclusion $\theta = \delta_2$

Conclusion :

Problem C : find $a(x)$ To minimize θ.
Problem B : find $a(x)$ To minimize δ_2

$\Longrightarrow$ Same problem $\theta = \delta_2$

Problem A $\longrightarrow$ Problem B $\longrightarrow$ Problem C.

$\downarrow$

Alternation Thm

F = Union of closed intervals.

Let $P(x)$ be an rth order polynomial.

$$P(x) = \sum_{k=0}^{r} a_k x^k$$

D = desired function continuous in F

W = positive fn

$$E(x) = W(x)\left[D(x) - P(x) \right]$$

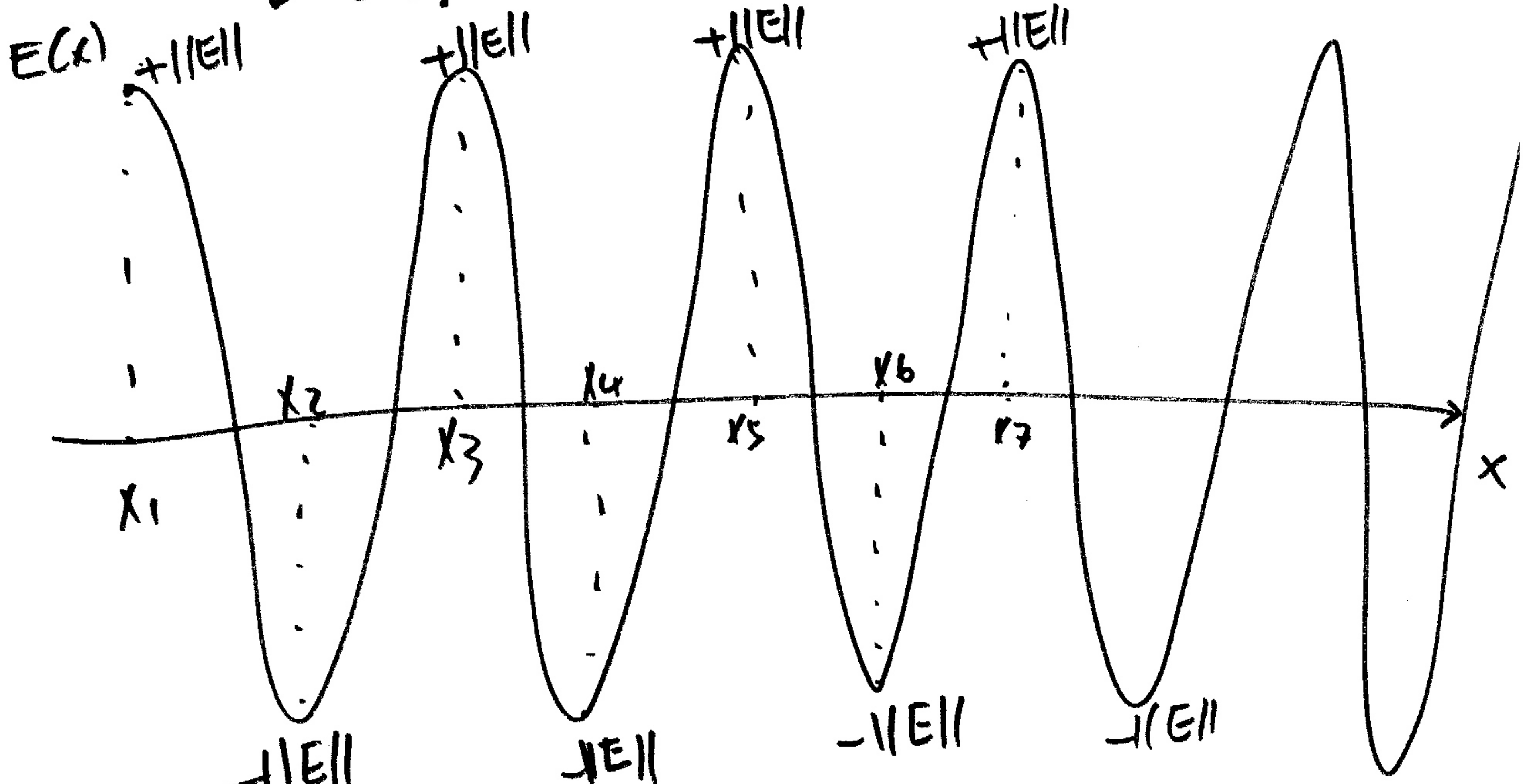
Define $$\|E\| = \max_{x \in F} |E(x)|$$

Necessary + sufficient condition for $P(x)$ to be a <u>unique</u> rth order polynomial that minimizes $\|E\|$ is that $E(x)$ exhibits <u>at least</u> $r+2$ alternations.

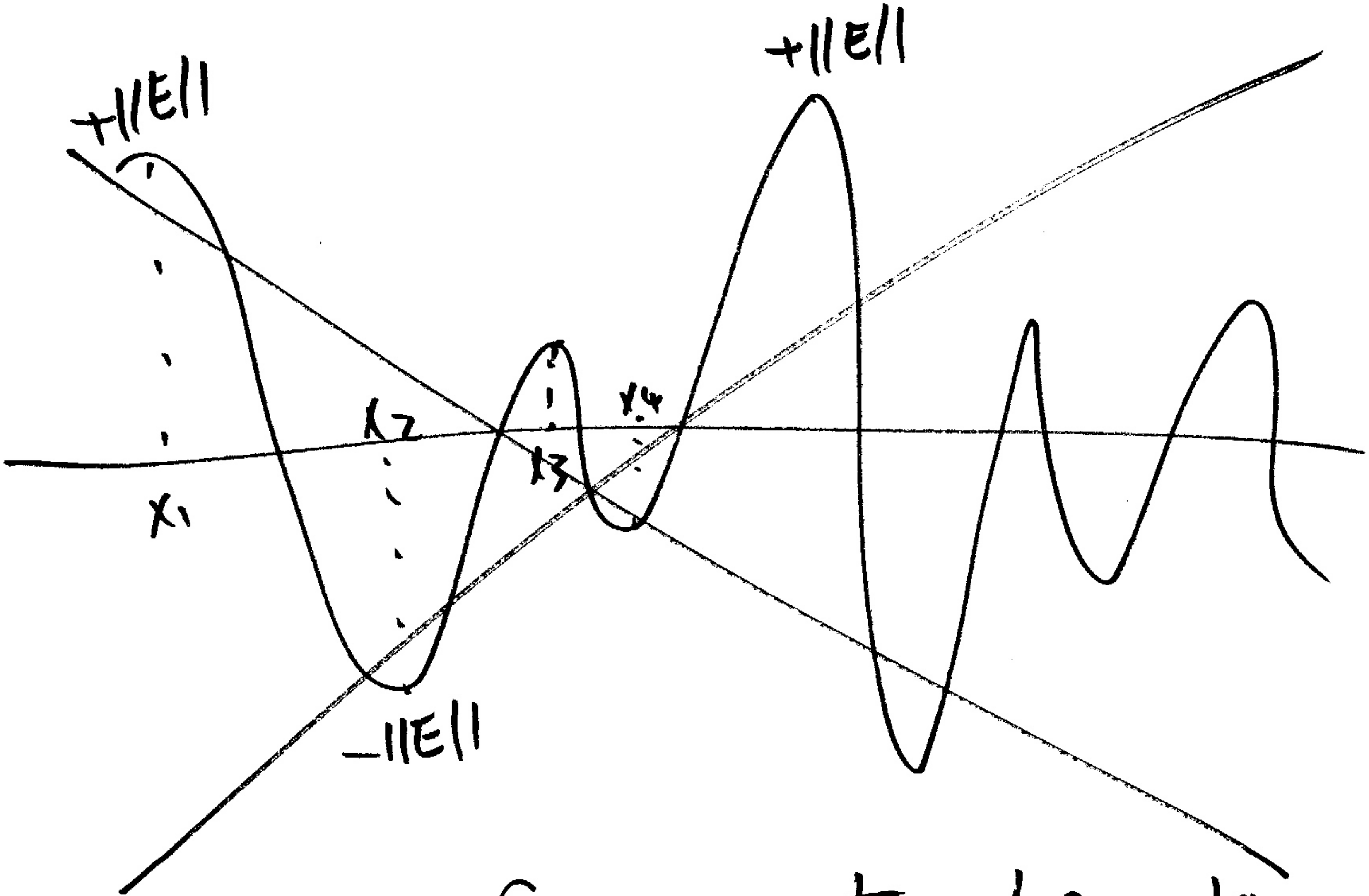
i.e. There is at least $r+2$ values of x

$$x_1 < x_2 < \cdots < x_{r+2}$$

$$E(x_i) = -E(x_{i+1}) = \pm\|E\|$$

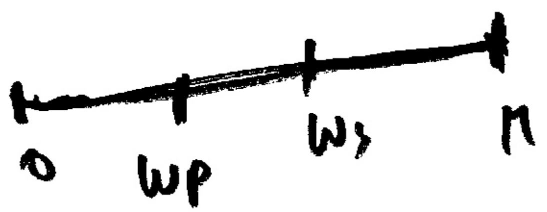


– – – – $r+2$ of them.



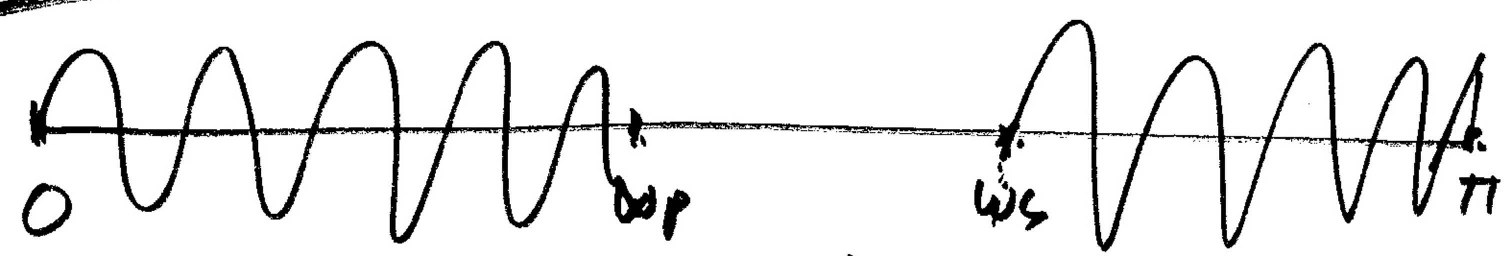
Cannot have this

Recall $G(\omega) = \sum_{n=0}^{M} a(n) \cos(\omega n)$



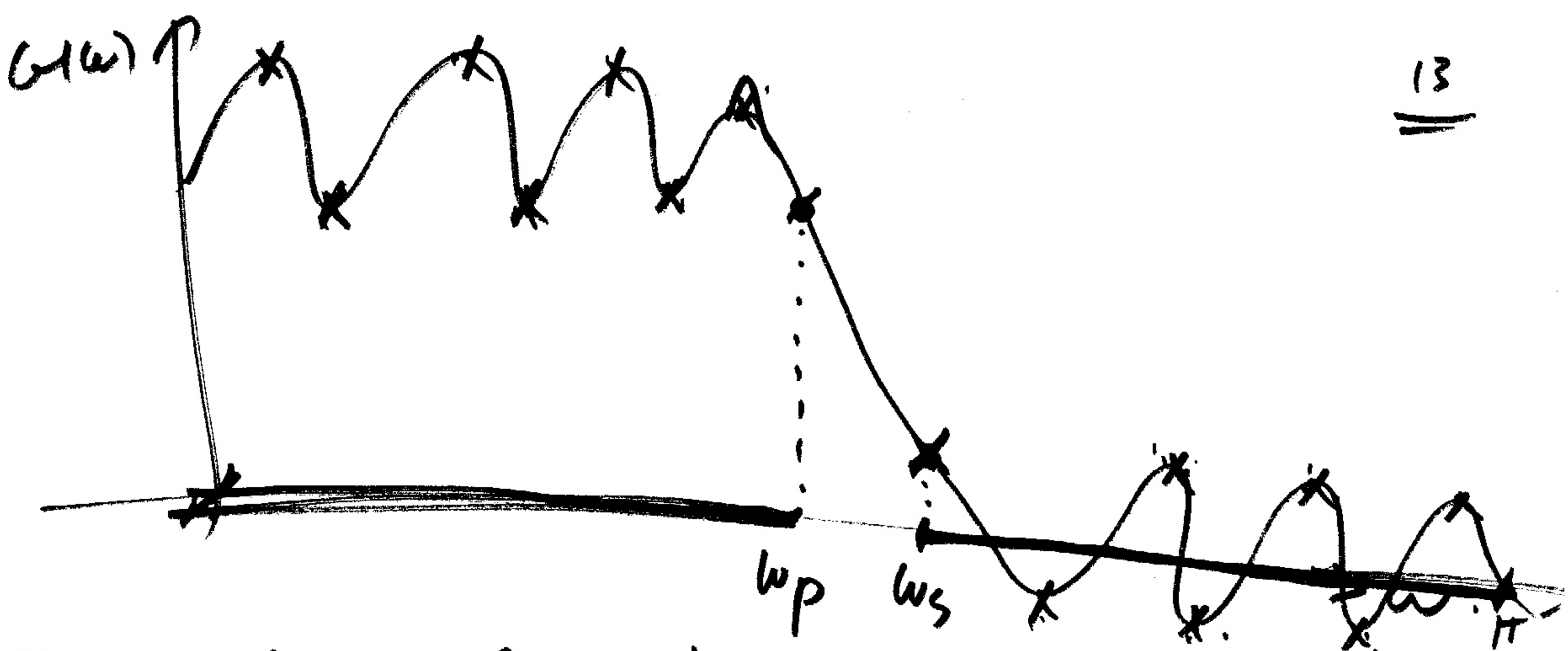
$\rightarrow$ Can have at most $M+1$ local extrema.

Alt. Thm Cares about $F = I_1 \cup I_2$



at least $M+2$ $(r+2)$ alternations in F.

even though ω_p and ω_s are not local extrema of $G(\omega)$.



Even though $G(\omega)$ between $0, \pi$ has at most $n+1$ local extrema, it can have up to $n+3$ local extrema in $I_1 \cup I_2 = F$

Alt Thm $I_1 \cup I_2 \Rightarrow$ at least $n+2$ alternation local extrema

$G(\omega)$ has either $n+2$ or $n+3$ local extrema in F that are also alternation

Fig 7.35, 7.37 O&S

Properties of optimal Filter

① For low pass filter, alternations always occur at ω_p and ω_s.

Fig 7.38.

$\Longrightarrow$ otherwise we "lose" 2 alternations

② Filter will be equiripple except possibly at $\omega = 0, \pi$

slope 0 or local extrema

Fig 7.39

Alg for optimum Filter Design

Parks/McClellan

Problem B: Given ω_s, ω_p, $K = \frac{\delta_1}{\delta_2}$

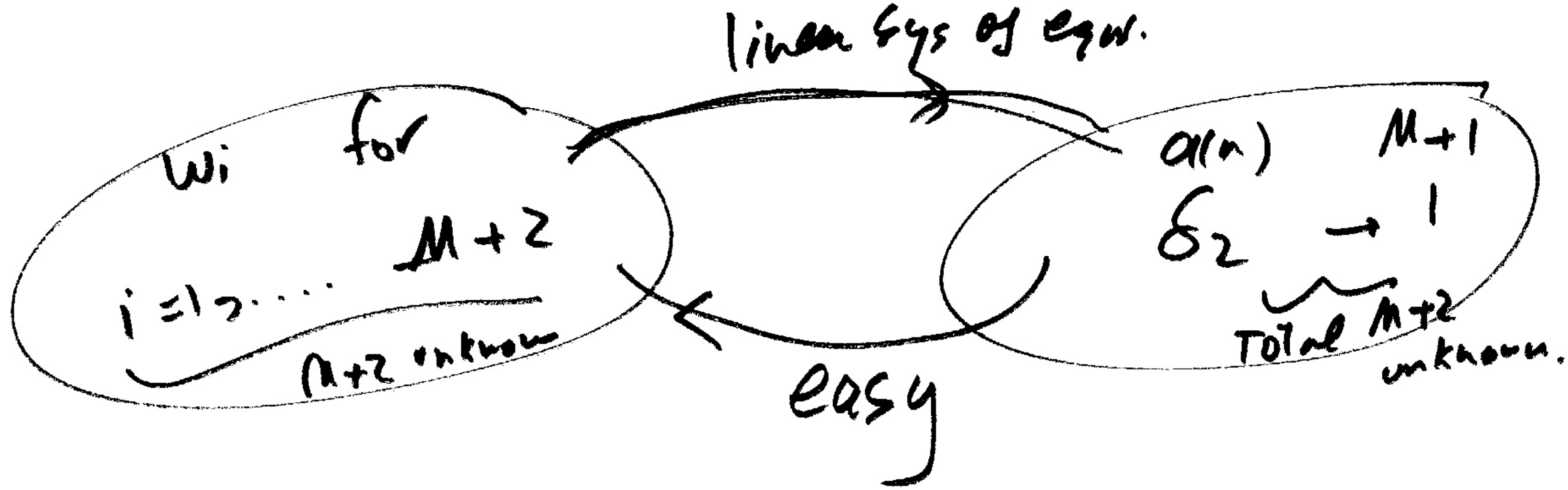
find $a(n)$ to minimize δ_2.

hitting ~~max~~ $\rightarrow$ $E(\omega_i) = \pm \delta_2 \qquad i = 1, \ldots, M+2$

alternating fashion $\rightarrow$ $E(\omega_i) = -E(\omega_{i+1}) \qquad i = 1, \ldots, M+1$

$$W(\omega_i)\left[G(\underline{\omega_i}) - D(\omega_i) \right] = (-1)^{i+1} \underline{\delta_2}$$

$$G(\omega) = \sum_{n=0}^{M} \underline{a(n)} \cos \omega n$$


linear sys of eqn.
w_i for
$i = 1, 2, \ldots, M+2$
$M+2$ unknown
$a(n)$ $M+1$
$\delta_2 \to 1$
Total $M+2$ unknown.
easy