Algorithms for Optimal Filter Design:

Problem B: Given w_s, w_p, $K = \frac{\delta_1}{\delta_2}$, M

find $a(n)$ to minimize δ_2

$$G(w) = \sum_{n=0}^{M} a(n) \cos(wn)$$

Alt Thm:

$$E(w_i) = \pm \delta_2 \qquad i = 1, \ldots, M+2$$

$$E(w_i) = -E(w_{i+1}) \qquad i = 1, \ldots, M+1$$

$$W(w_i)\left[G(w_i) - D(w_i)\right] = (-1)^{i+1} \delta_2.$$

$$\begin{tikzcd}
\begin{matrix} \text{local extrema} \\ w_i \\ i=1,\dots,M+2 \end{matrix} \arrow[r, bend left, "\text{harder}"] & \begin{matrix} a(n) \to M+1 \\ \delta_2 \to 1 \end{matrix} \arrow[l, bend left, "\text{easy}"]
\end{tikzcd}$$

"harder" show if we know w_i, how to find $a(n)$ and δ_2?

$$\left.\begin{aligned} G(w_i) &= \sum_{n=0}^{M} a(n) \cos(w_i n) \\ G(w_i) &= \frac{(-1)^{i+1}\delta_2}{W(w_i)} + D(w_i) \end{aligned}\right\} \Longrightarrow$$

$$\sum_{n=0}^{M} a(n) \cos(w_i n) = \frac{(-1)^{i+1}\delta_2}{W(w_i)} + D(w_i) \qquad i=1,\dots,M+2$$

$\longrightarrow$ linear in $a(n)$ and in δ_2; ie linear system of equations

$i=1$

$$a(0)\cos(\omega_1 \cdot 0) + a(1)\cos(\omega_1) + a(2)\cos(2\omega_1) + \dots a(M)\cos(M\omega_1) = \frac{(-1)^{1+1}\delta_2}{W(\omega_1)} + D(\omega_1)$$

$\vdots$

$i=2$ $\quad a(0)\cos(\omega_2 \cdot 0) + a(1)\cos(\omega_2) \dots$

$\vdots$

$i = M+2$

Eqn is linear in both $a()$ and in δ_2

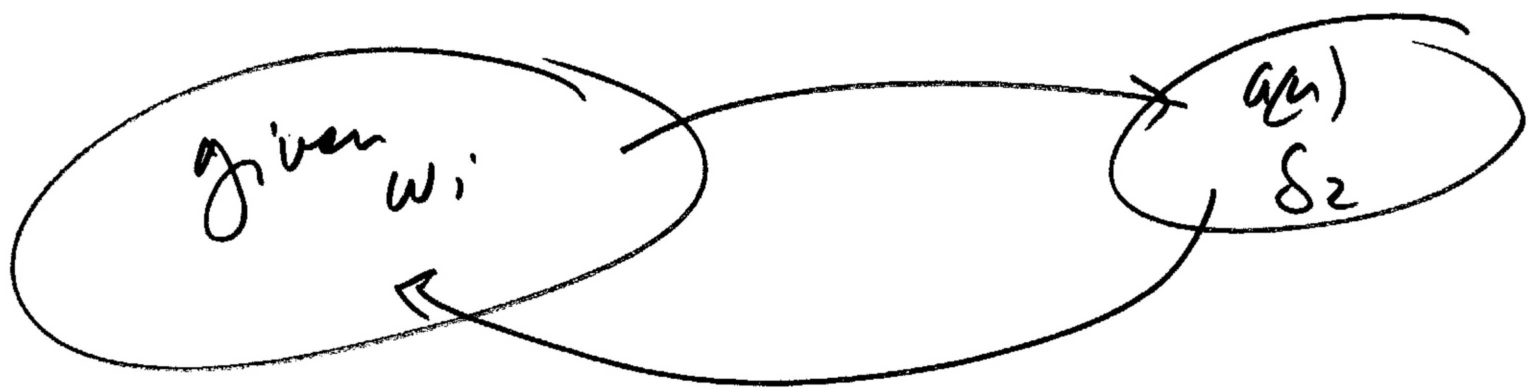
$M+2$ eqns + $M+2$ unknown $\Longrightarrow$ get answer

$$\underbrace{\begin{bmatrix} \cos(w_1 \cdot 0) & \cos w_1 & \cos 2w_1 & \cdots & \cos w_1 M \\ \cos 0 & \cos w_2 & \cos 2w_2 & \cdots & \cos M w_2 \\ \cos 0 & \cos w_3 & \cos 2w_3 & \cdots & \cos M w_3 \\ \vdots & & & & \\ \cos 0 & \cos(w_{M+2}) & \cos(2 w_{M+2}) & \cdots & \end{bmatrix}}_{\text{if } w_i \text{ known entries in this matrix is also entirely known}} \underbrace{\begin{bmatrix} a(0) \\ a(1) \\ a(2) \\ \vdots \\ a(M) \end{bmatrix}}_{\text{unknown}} = \underbrace{\delta_2}_{\text{unknown}} \underbrace{\begin{bmatrix} \frac{(-1)^{1+1}}{W(w_1)} \\ \frac{(-1)^{2+1}}{W(w_2)} \\ \vdots \\ \frac{(-1)^{M+2+1}}{W(w_{M+2})} \end{bmatrix}}_{\text{known}} + \underbrace{\begin{bmatrix} D(w_1) \\ D(w_2) \\ \vdots \\ D(w_{M+2}) \end{bmatrix}}_{\text{known}}$$

$$\underline{A \vec{X} = \vec{b}}$$

$$
\begin{bmatrix}
\cos 0 & \cos w_1 & \cdots & \cos M w_1 & -\frac{(-1)^{1+1}}{W(w_1)} \\
\cos 0 & \cos w_2 & \cdots & \cdots & \cdots \\
\cos 0 & \cos w_3 & \cdots & \cos M w_3 & -\frac{(-1)^{2+1}}{W(w_2)} \\
\vdots & & & & \\
\cos 0 & \cos w_{M+2} & \cdots & \cdots & -\frac{(-1)^{M+2+1}}{W(w_{M+2})}
\end{bmatrix}
\begin{bmatrix}
a(0) \\ a(1) \\ \vdots \\ a(M) \\ \delta_2
\end{bmatrix}
=
\begin{bmatrix}
D(w_1) \\ D(w_2) \\ \vdots \\ D(w_{M+2})
\end{bmatrix}
$$

Remez Exchange Algorithm.



Parks + McClellan showed: if ω_i are known, δ_2 is given by the following expression

$$\delta_2 = \frac{\displaystyle\sum_{k=1}^{M+2} b_k\, D(\omega_k)}{\displaystyle\sum_{k=1}^{M+2} \frac{b_k\,(-1)^{k+1}}{W(\omega_k)}} \qquad b_k = \prod_{\substack{i=1 \\ i\neq k}}^{M+2} \frac{1}{\cos\omega_k - \cos\omega_i}$$

Empirical Studies

Appro length of filter $\simeq 1 - \dfrac{10 \log_{10}(\delta_1 \delta_2) - 13}{2.3 \quad \Delta\omega}$.

$\Delta\omega = \omega_s - \omega_p$.

Kaiser : window length $\simeq 1 + \dfrac{A - 8}{2.2\Delta\omega}$ $\quad A = -20\log_{10}\delta$

Ex $\quad \omega_p = 0.4\pi \quad \omega_s = 0.6\pi \quad \delta_1 = 0.01$
$\delta_2 = 0.001$

optimum filter $\longrightarrow$ # of Tap $= 2M+1 = 27$

Kaiser Window $\longrightarrow$ # of Tap $\simeq 38$

Show 7.43

IIR Filter Design

- Interested in IIR filters That have Rational Transfer fns.

$$H(z) = \frac{P(z)}{Q(z)}$$

polynomials in z.

$$\left.\begin{aligned} P(z) &= \sum_{n=0}^{N} \underline{a(n)}\, z^{n} \\ Q(z) &= \sum_{n} \underline{b(n)}\, z^{n} \end{aligned}\right\}$$

Filter Design is To find $a(n)$ and $b(n)$

IIR filter Design

Transformation

Direct numerical Technique

impulse Invariance

Bilinear.

Transformation

1. Given set of Discrete Time Digital Filter spec
2. Transform specs from Discrete Time to Continuous time.

   $z \longrightarrow s$.
3. Design Continuous time IIR (analog).

   $H_a(s)$ ← analog.
4. $H_a(s) \xrightarrow{\text{Transform}} H(z)$

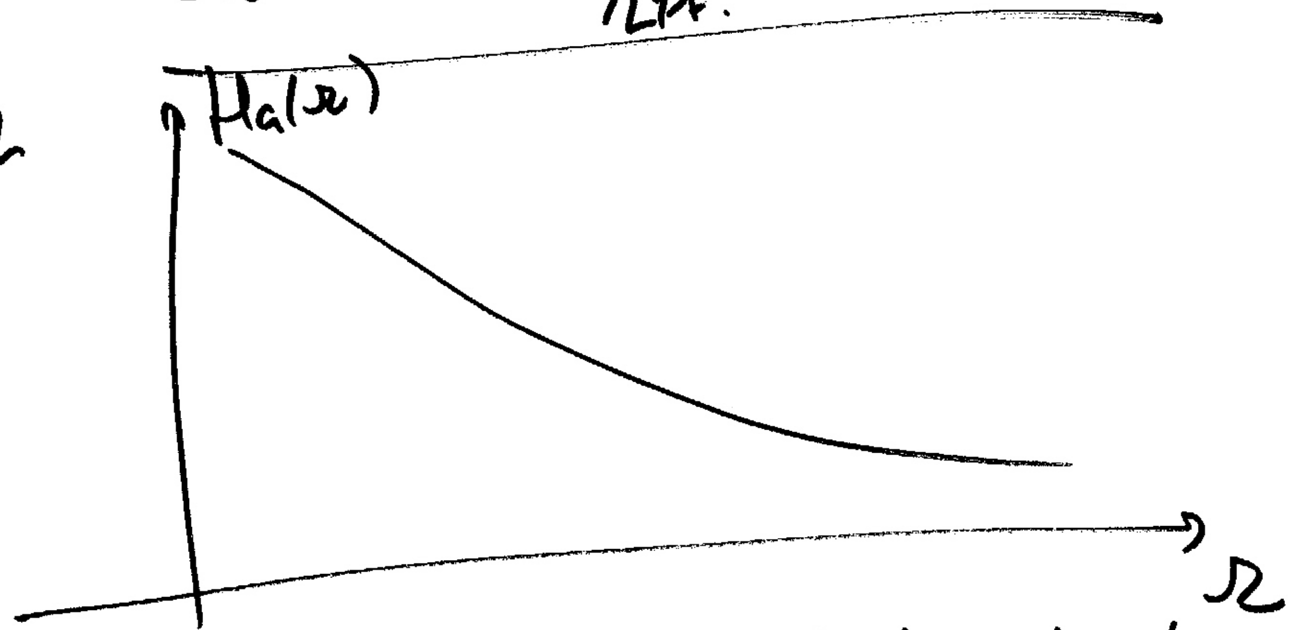
   Continuous time $\longrightarrow$ Discrete Time.

   $s \longrightarrow z$

Continuous time IIR filters
LPF.

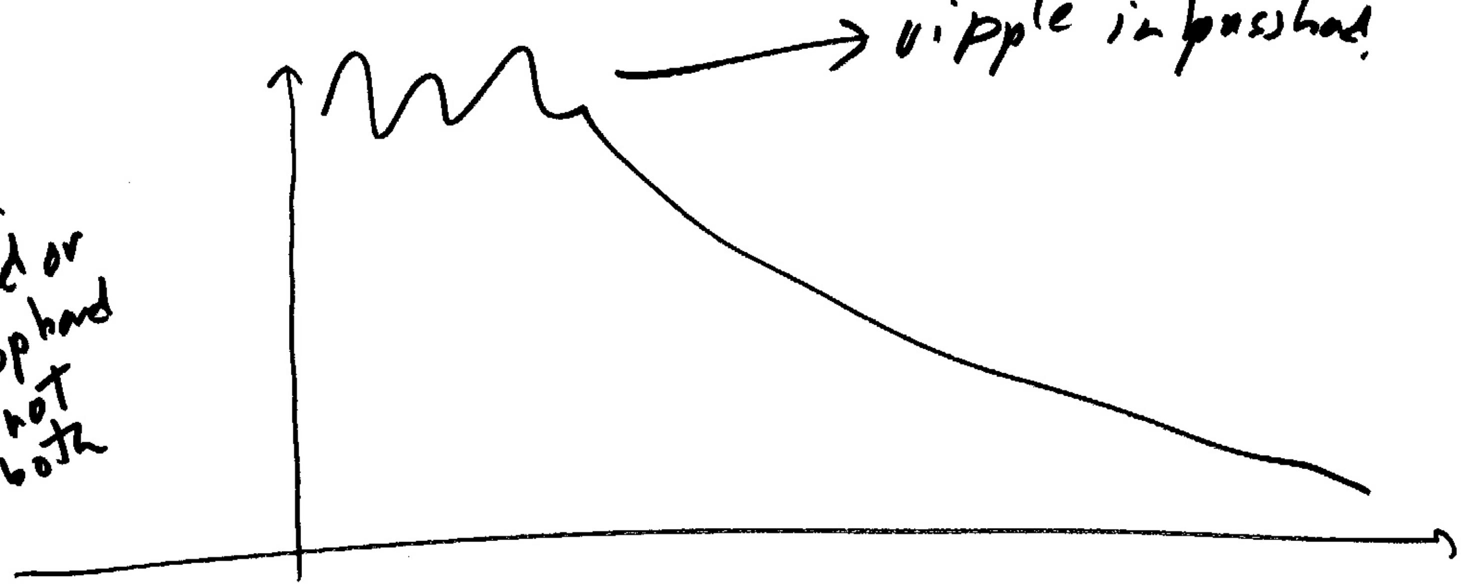
1. Butterworth

monotonic in passband + stopband

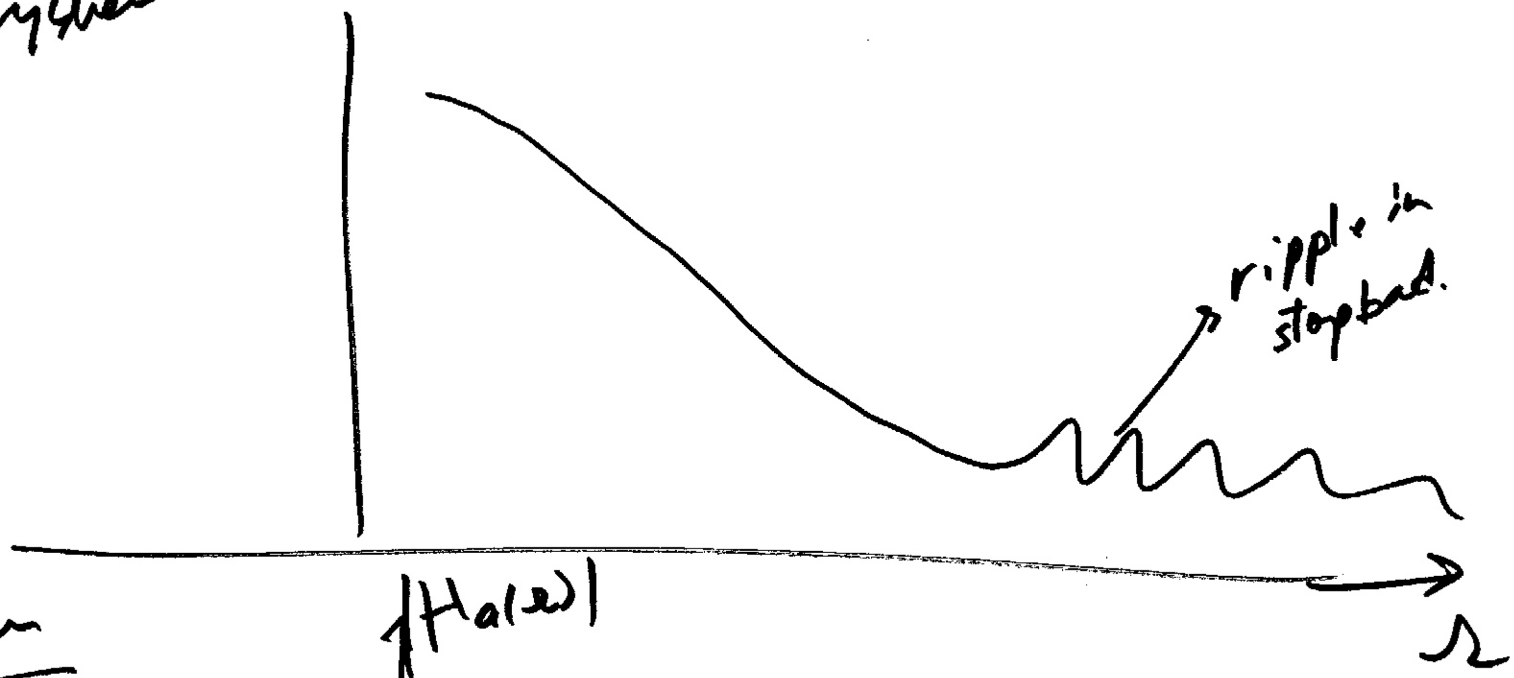


~~Chebychep~~
Chebyshev

ripple in passband or in stopband But not in both

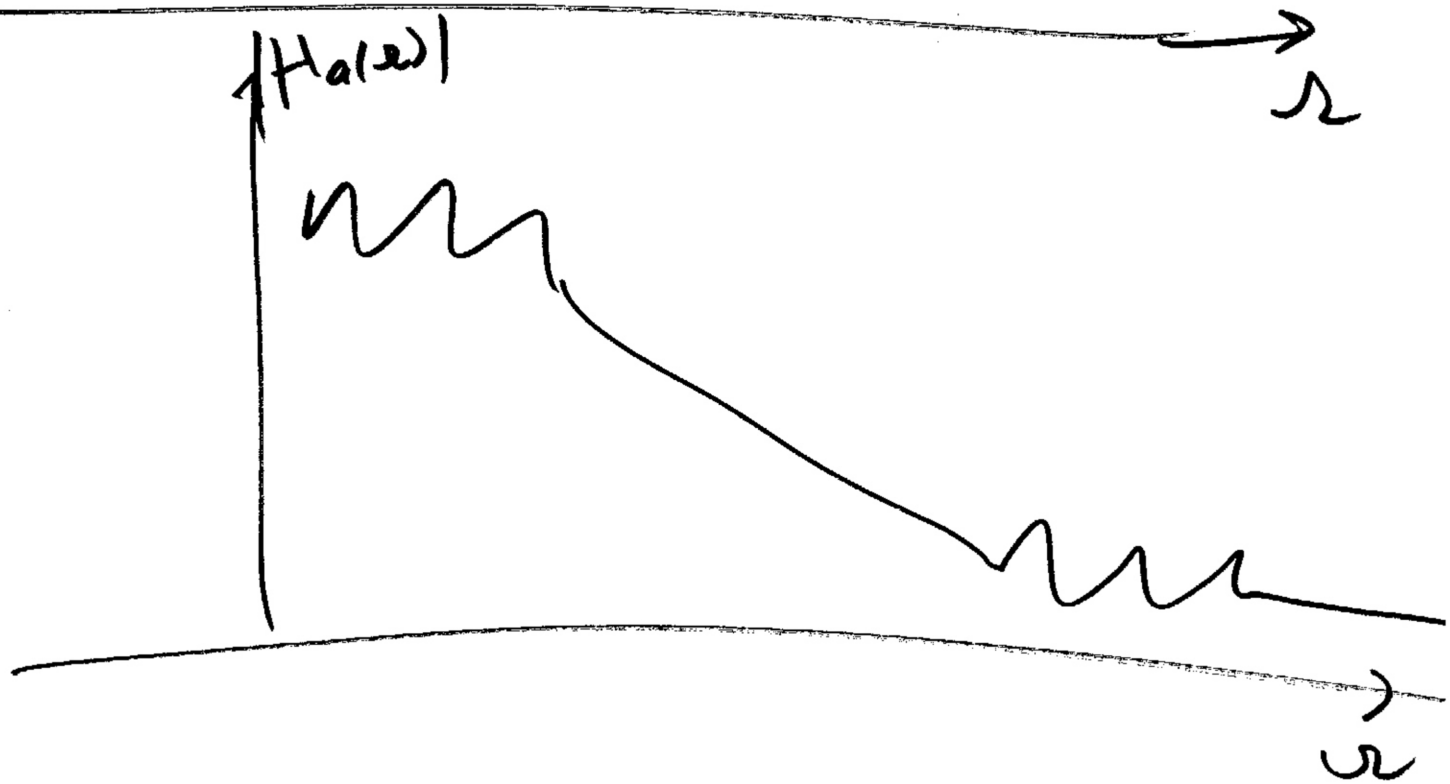


Chebyshev



Elliptic filter

ripple in passband and in stopband



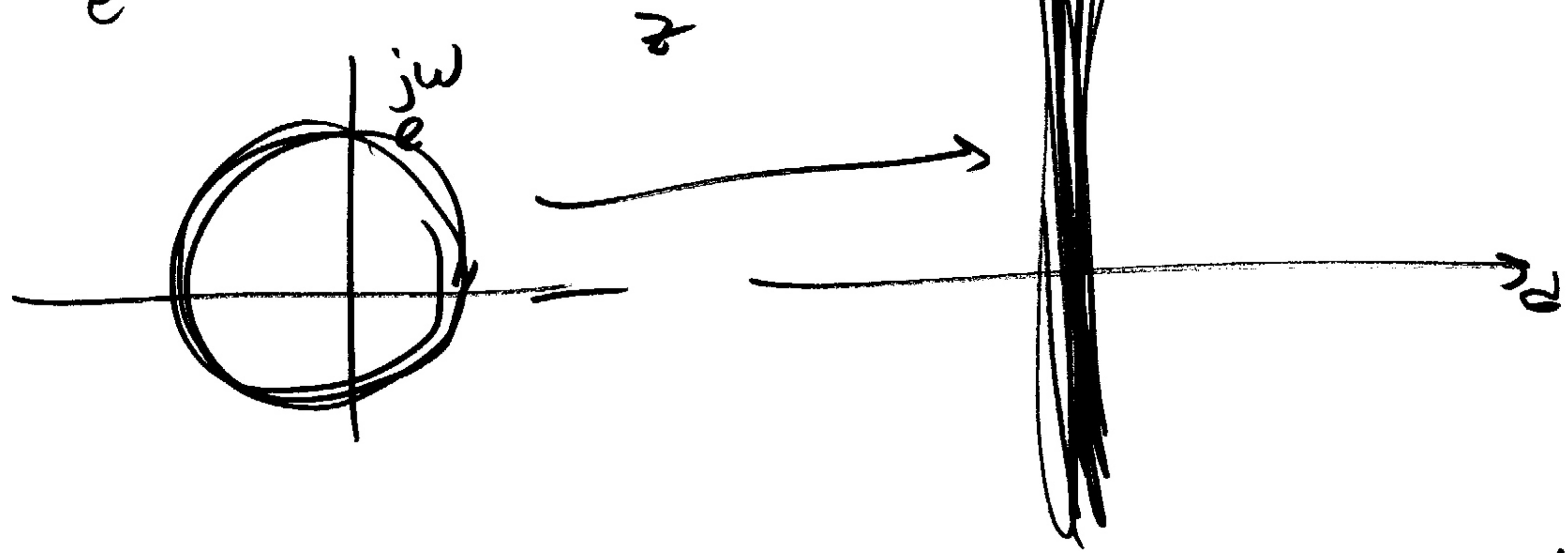
Desirable Properties of Transformation

① Causal + stable analog filter to get Transformed into causal + stable discrete time filter.

$$H_a(s) \xrightarrow{\text{xform}} H(z)$$

$H_a(s)$: causal + stable $\qquad$ $H(z)$: causal + stable

② $j\Omega$ axis in s plane to get transformed onto $e^{j\omega}$ circle in z plane



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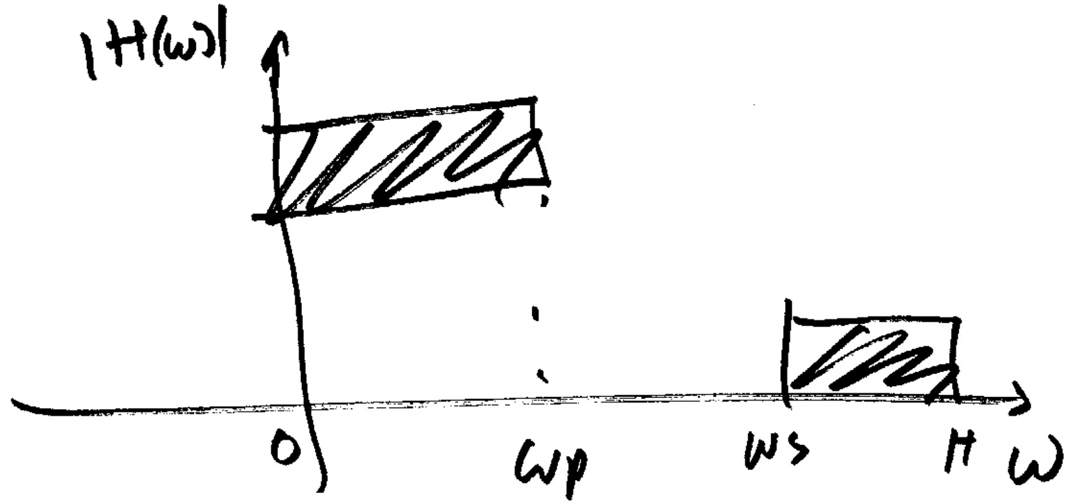

Need This To Translate specs from

Discrete $\longrightarrow$ Analog.

$e^{j\omega}$ $\longrightarrow$ $j\Omega$

z plane $\qquad$ s plane.

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(3) Rational $H_a(s)$ $\longrightarrow$ Rational $H(z)$.

analog $\qquad$ Discrete

why? Implementation using Diff. eqns.

~~$z = \sqrt{s}$~~ $\quad$ ~~$s = \sqrt{z}$~~