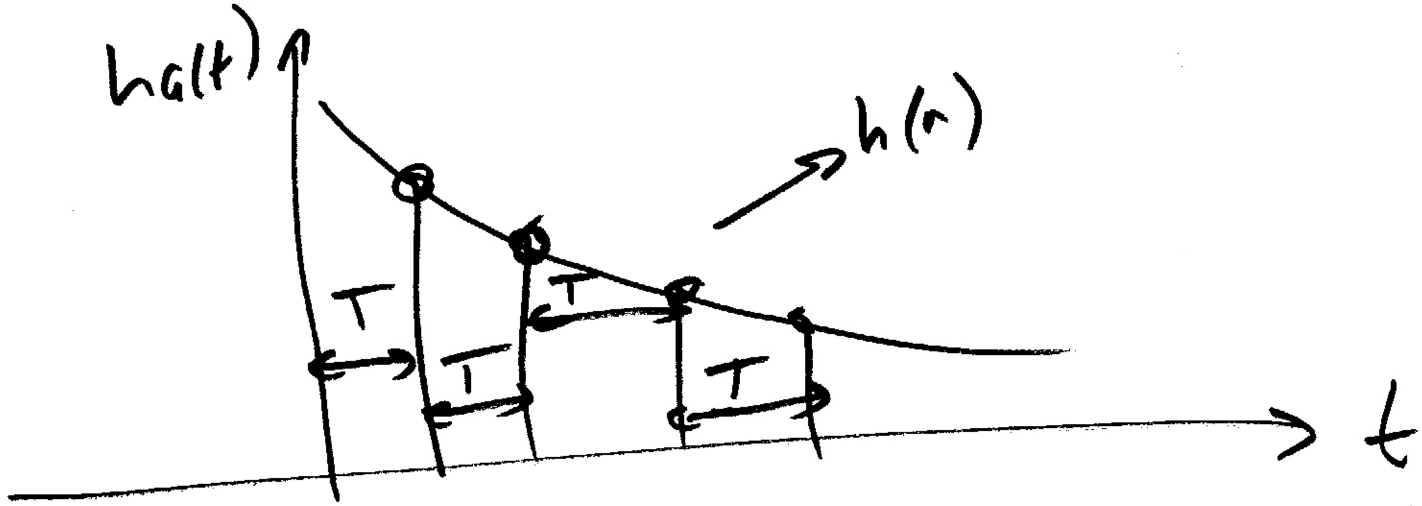
Impulse Invariance Transformation

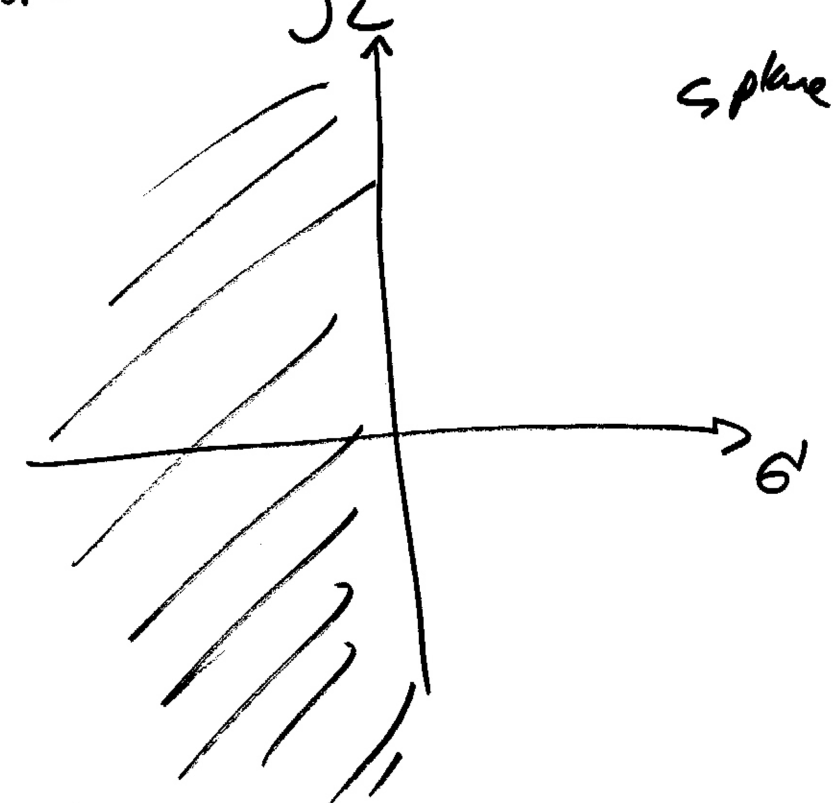
$$H_a(s) \rightarrow h_a(t) \longrightarrow h(n) = \left[h_a(t)\right]_{t=nT} \rightarrow H(z)$$



Does this satisfy all 3 Desirable properties?

① Does causal + stable $H_a(s)$ Transform onto causal + stable $H(z)$?

assume causal + stable



$$H_a(s) = \sum_k \frac{A_k}{s - s_k}$$

causality + stability $\Longrightarrow$ poles have to be in Left half plane.

$\Longrightarrow Re[s_k] < 0$

$$h_a(t) = \sum_k A_k e^{s_k t} u(t)$$

exploit causality

Transformation.

$$h(n) = \left[h_a(t)\right]_{t=nT} = \sum_K A_K \, e^{s_K nT} \, \underline{u(n)}$$

This is causal. ✓

$$H(z) = \sum_K A_K \frac{1}{1 - e^{s_K T} z^{-1}}$$

Q is this filter stable?

A is poles inside unit circle?

$$|e^{s_K T}| < 1 \;??$$

$$Re[s_k] < 0 \Longrightarrow \left| e^{s_k T} \right| < 1 \Longrightarrow$$

poles are inside unit circle.

$$\left| e^{s_k T} \right| = \left| e^{Re[s_k] T} \cdot e^{j Im[s_k] T} \right|$$

$$= \left| e^{Re[s_k] T} \right| < 1$$

$\uparrow$

$Re[s_k] < 0$

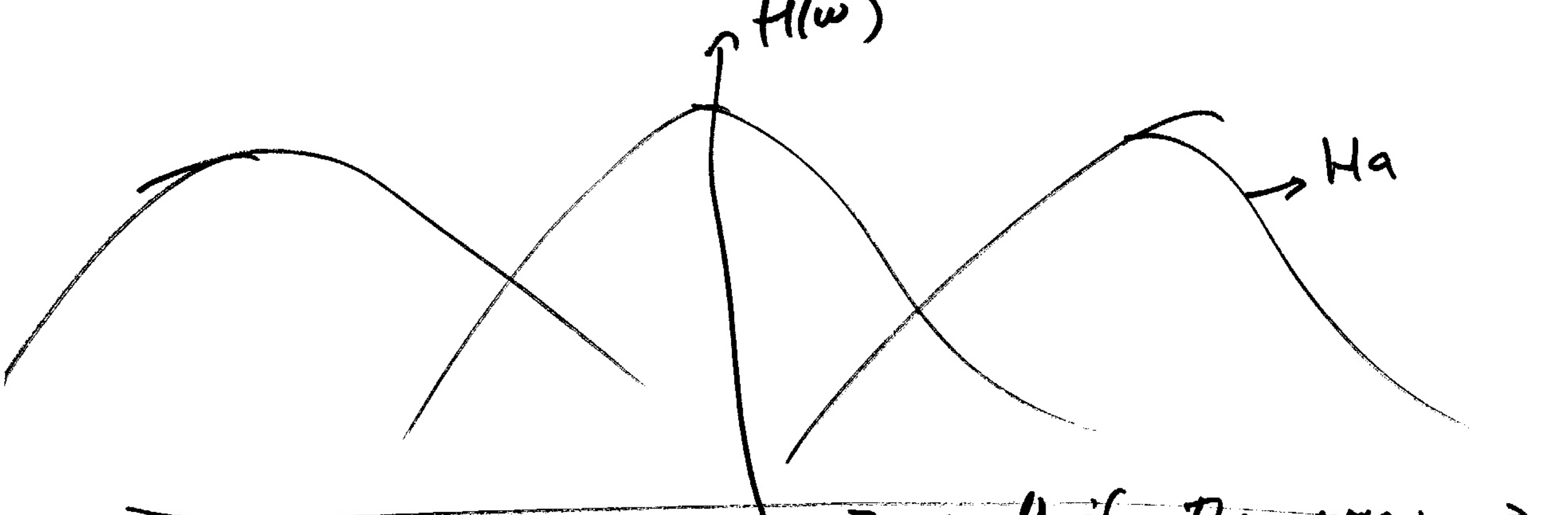
→ Guaranteed Causal + stable $H_a(s)$ ⟶ causal + stable $H(z)$

→ Also guaranteed Rational $H_a(s)$ ⟶ Rational $H(z)$

① Does $j\Omega$ axis in s plane $\longrightarrow$ e^{jw} circle in z plane?

$$h(n) = \left[h_a(t) \right]_{t=nT}$$

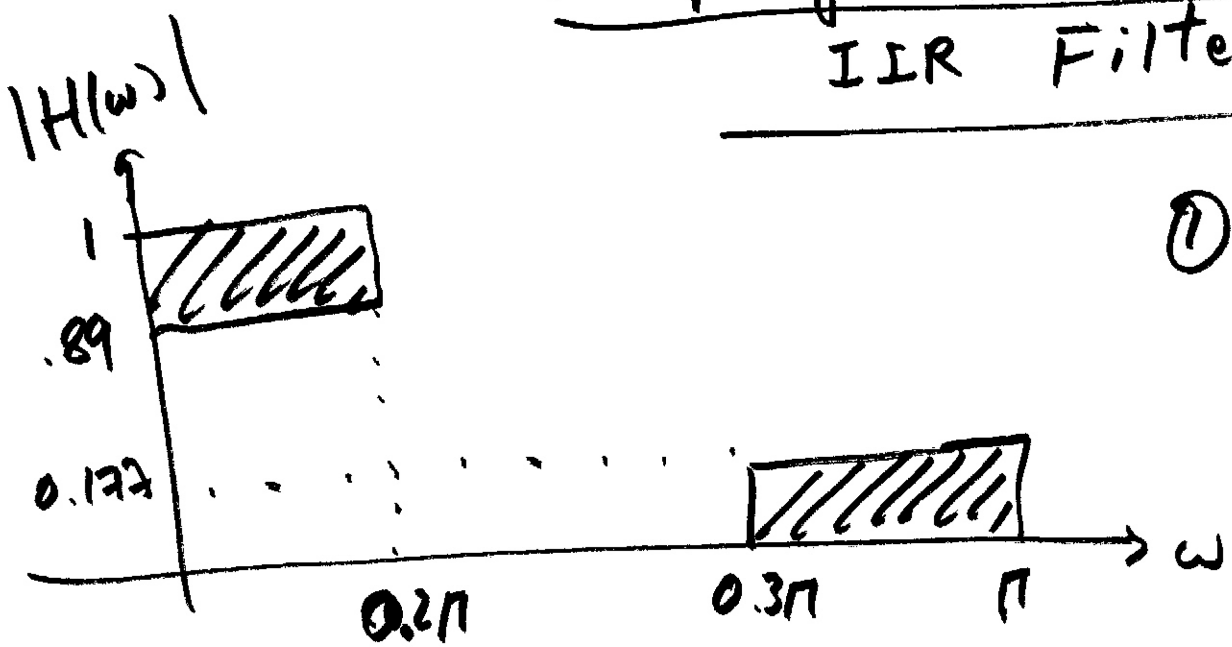
$$H(\omega) = \frac{1}{T} \sum_r H_a\left(\frac{\omega}{T} - \frac{2\pi r}{T} \right)$$



Can I translate the spec? could if there was no aliasing

Example of "Impulse Invariance Transformation"

IIR Filter Design



① Given specs in D.T. Domain

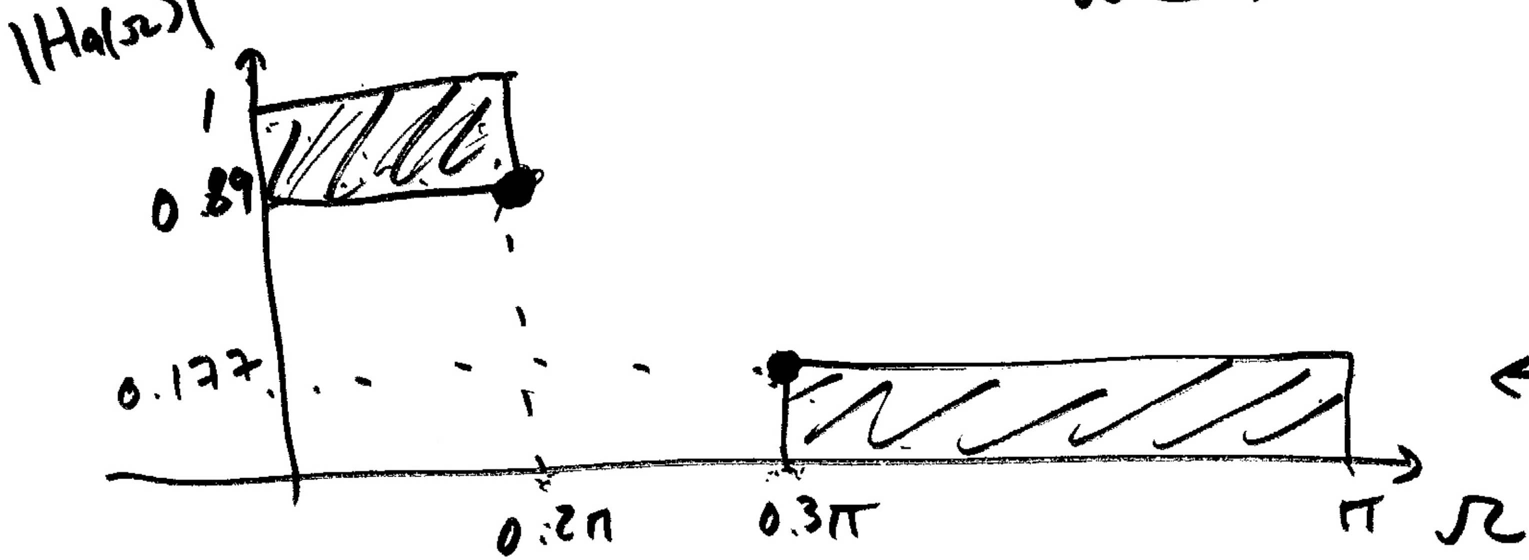
$0.89 \leq |H(\omega)| \leq 1 \qquad 0 \leq |\omega| \leq 0.2\pi$

$|H(\omega)| < 0.177 \qquad 0.3\pi < |\omega| < \pi$

step ② Transform D.T. specs into C.T. specs.

choose $T = 1$

$\omega \longrightarrow \Omega$.



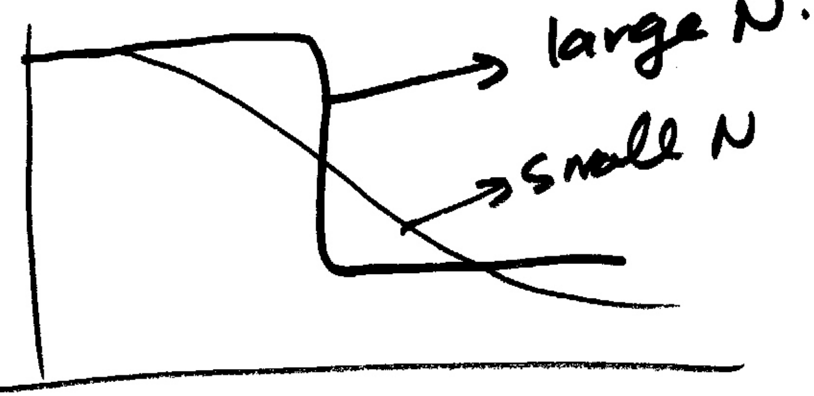
③ Design C.T. filter that satisfies specs in

Approach: Butterworth filter.

Tutorial on Butterworth filter of order N :

$$|Ha(\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad *$$

Ω_c = cut off frequency.   N = order.



To minimize the order of the resulting filter.
Let us impose the following constraints

$$\left\{ \begin{aligned} |H_a(\Omega)|_{\Omega=0.2\pi} &= 0.89 \\ |H_a(\Omega)|_{\Omega=0.3\pi} &= 0.177 \end{aligned} \right.$$

Combine with eqn * To compute N, Ω_c

eqn ① $$|H_a(\Omega)|^2_{0.2\pi=\Omega} = (0.89)^2 = \frac{1}{1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N}}$$

eqn ② $$|H_a(\Omega)|^2_{\Omega=0.3\pi} = (0.177)^2 = \frac{1}{1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N}}$$

2 eqns, un known $\Longrightarrow \left\{ \begin{aligned} &N = 5.88 \rightarrow N = 6 \\ &\Omega_c = 0.704 \end{aligned} \right.$

Round $N = 5.88 \rightarrow$ to $N = 6$. To have integer as order

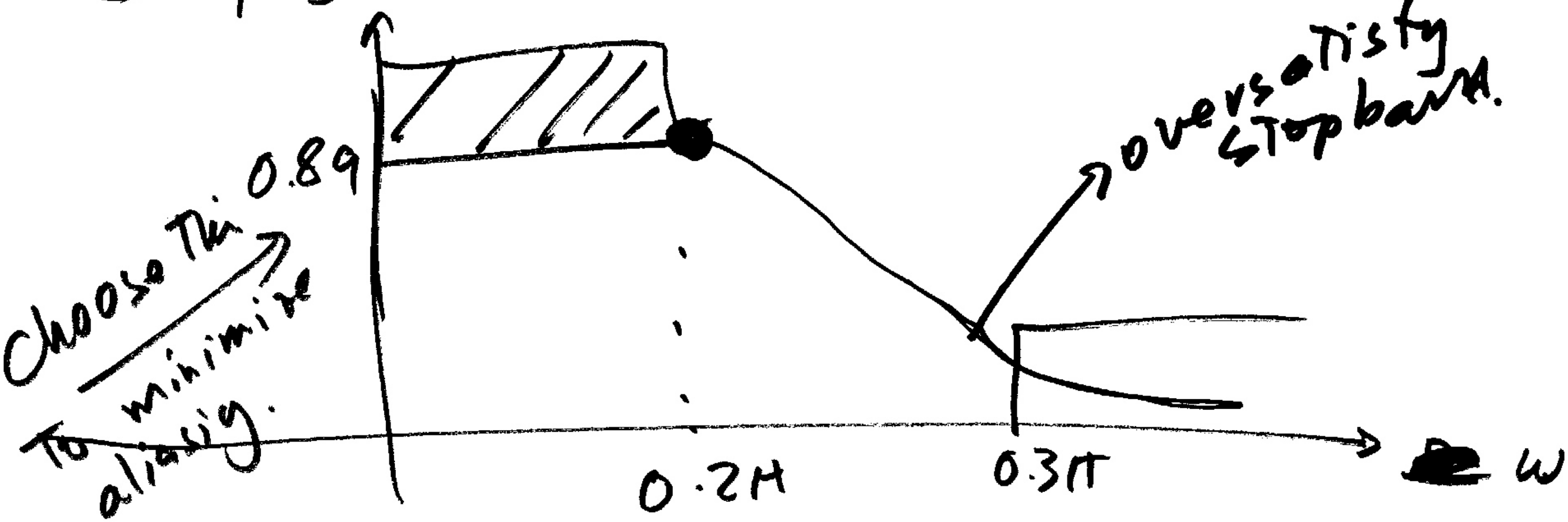
Must change $\Omega_c \rightarrow \Omega_c'$

<u>Question</u>: Which eqn should I use To re-compute Ω_c' given $N = 6$?

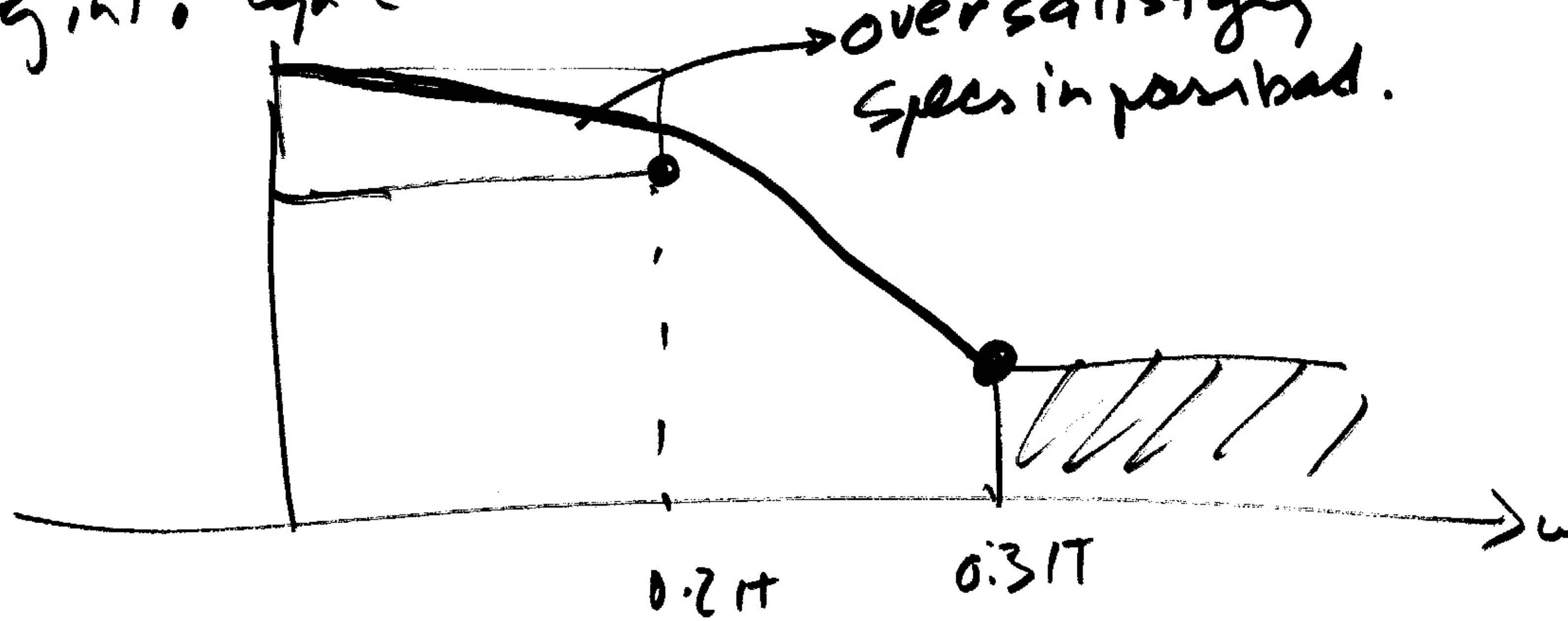
$\Rightarrow$ Two choices: (1) plug into eqn To To recompute Ω_c

(2) plug into eqn 2

If plug into Eqn 1



plug into eqn 2



eqn ① let $N=6$

$$(0.89)^2 = \frac{1}{1+\left(\frac{0.2\pi}{\Omega_c}\right)^{12}}$$

$$\Rightarrow \Omega_c = 0.7032$$

Done with Butterworth Analog filter Design.

$\Omega_c = 0.7032$

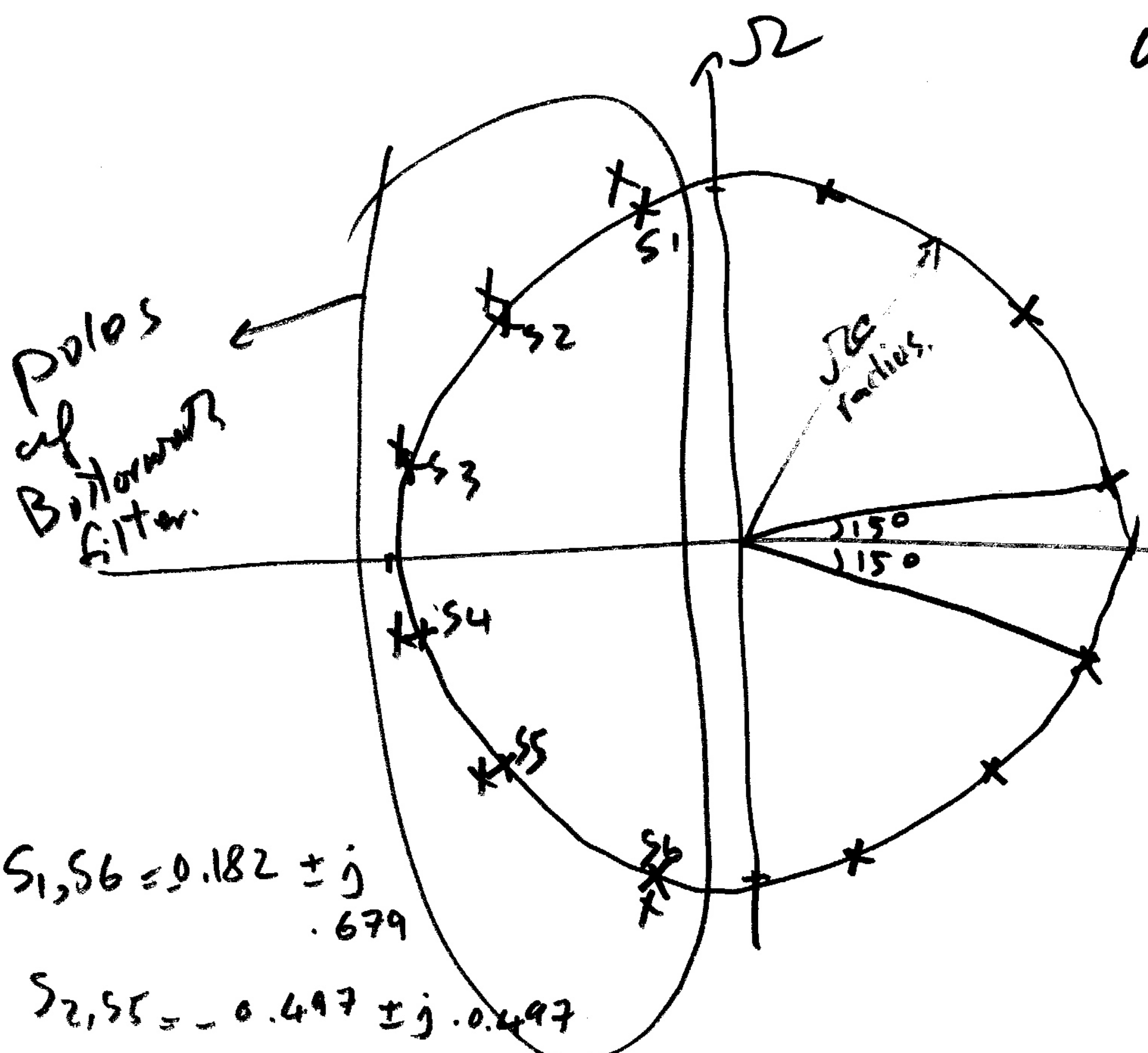
$N = 6$

$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{0.7032}\right)^{12}}$$

$$H_a(s)\,H_a(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

angle between poles $= \frac{360}{12} = 30^\circ$

$\frac{360}{2N} =$



s plane

$S_1, S_6 = 0.182 \pm j\,.679$

$S_2, S_5 = -0.497 \pm j\,0.497$

$S_3, S_4 = -0.679 \pm j\,0.182$

$$H_a(s) = \frac{0.12093}{(s^2+0.3645s+0.494)(s^2+0.9455s+0.494)(s^2+1.358s+0.496)}$$

4 Back To D.T. Domain. $\boxed{T=1}$

$$H_a(s) \longrightarrow H(z)$$

$$H_a(s) = \sum_k \frac{A_k}{s-s_k} \rightarrow H(z) = \sum_k \frac{A_k}{1-e^{s_k T} z^{-1}}$$

$$H(z) = \frac{.2871 - 0.4466z^{-1}}{1-1.2971z^{-1}+0.9444z^{-2}} + \frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3966z^{-2}}$$

$$+ \frac{1.8557 - 0.6303z^{-1}}{1-0.9972z^{-1}+0.257z^{-2}}$$

Fig 7.5 OKS

Bilinear Transformation

$$H_d(z) = \Big[H_a(s) \Big]_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

discrete

* Since $\frac{1-z^{-1}}{1+z^{-1}}$ is rational then.

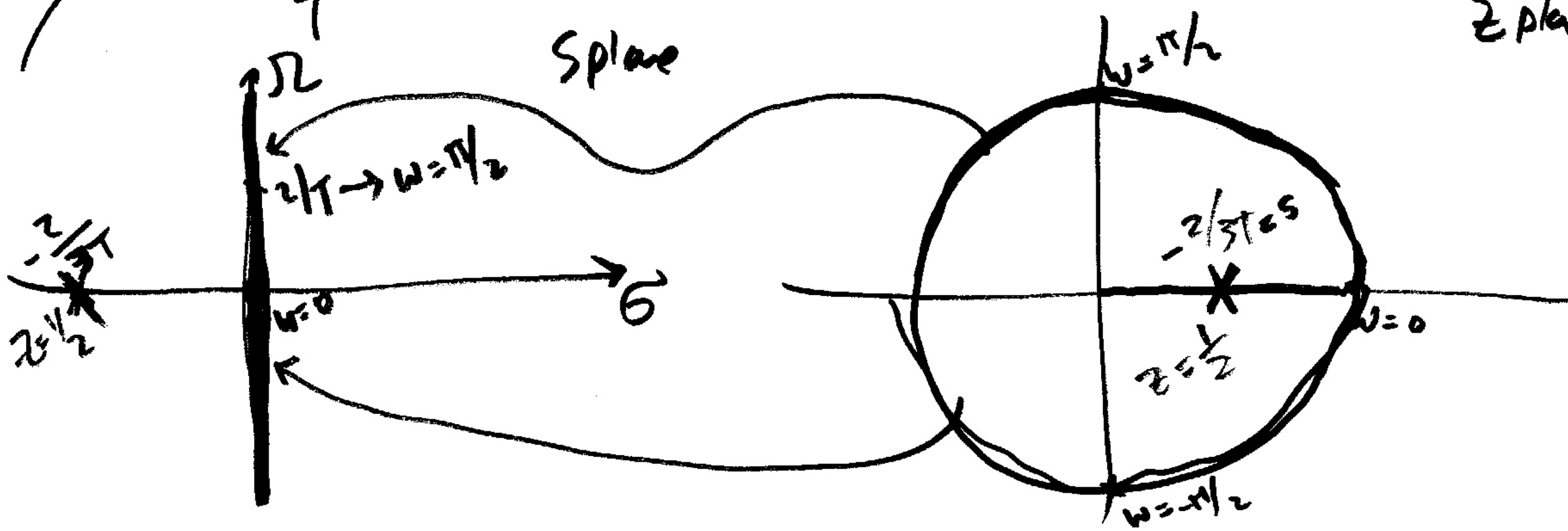
  Rational Transfer fn in C.T. $\longrightarrow$ Rational D.T. Transfer fn.

* Does $j\Omega$ axis (s plane) get mapped onto $e^{j\omega}$ circle? (z plane).

Let $z = e^{jw} \Rightarrow S = \frac{2}{T}\left(\frac{1-e^{-jw}}{1+e^{-jw}}\right)$

$$S = \frac{2}{T}\,\frac{e^{-jw/2}\left(e^{jw/2} - e^{-jw/2}\right)}{e^{-jw/2}\left(e^{jw/2} + e^{-jw/2}\right)}$$

$$S = \frac{2}{T}\, j \tan\frac{w}{2} = \sigma + j\,\Omega \Rightarrow$$



$$\begin{cases} \sigma = 0 \\ \Omega = \dfrac{2}{T} \tan \dfrac{w}{2} \end{cases}$$

As $w: 0 \longrightarrow \pi$
$\Omega: 0 \longrightarrow \infty$

As $w: 0 \longrightarrow -\pi$
$\Omega: 0 \longrightarrow -\infty$

① Does causal stable $H_a(s) \longrightarrow$ causal + stable $H(z)$?

Need to show LHP in s domain is mapped into <u>INSIDE</u> unit circle in z domain.

— only need to show this point.
(since transformation w, Ω is smooth, cont. diff.)

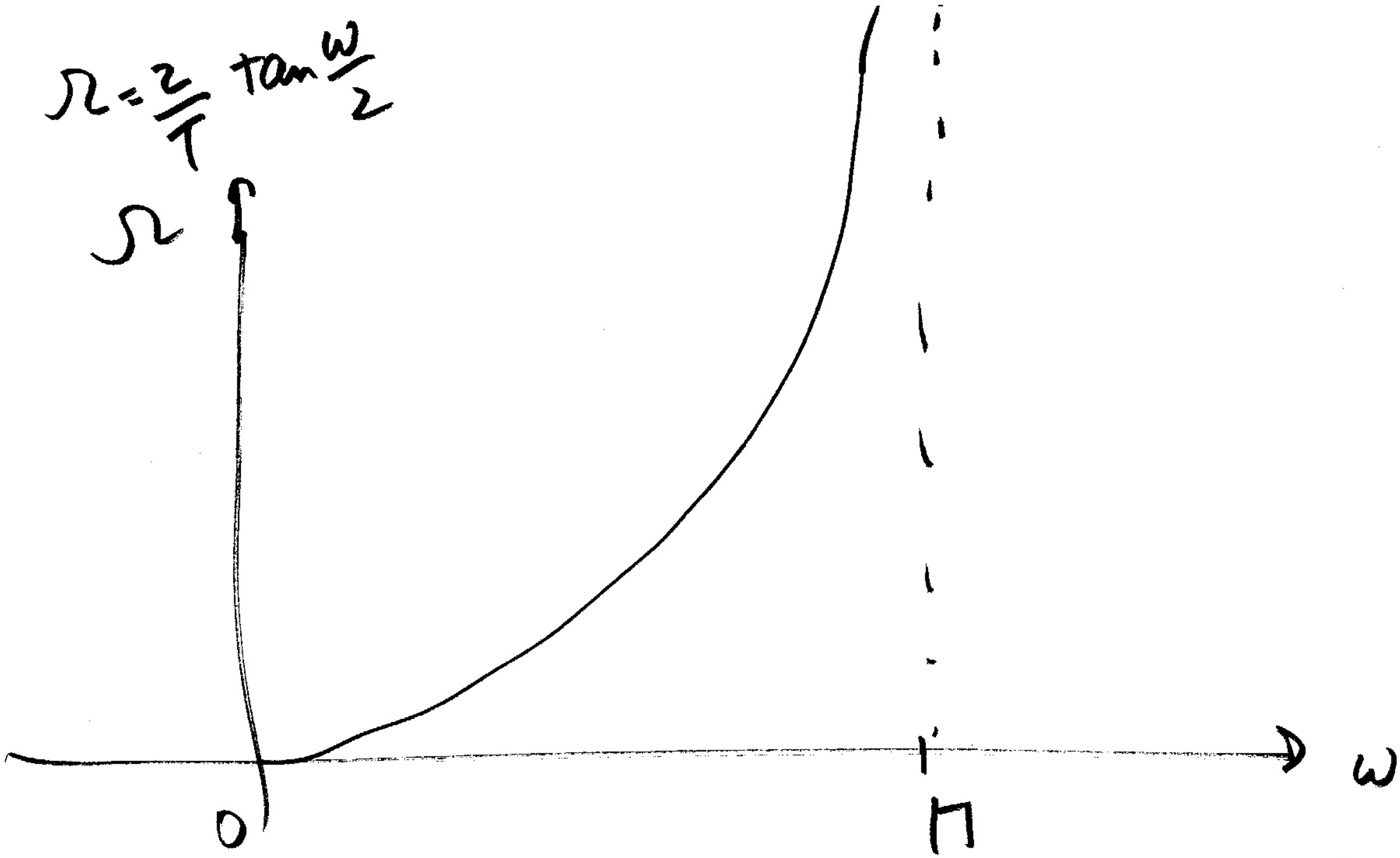
pick $s = -\frac{2}{3T}$

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = -\frac{2}{3T}$$

$\Rightarrow z = \frac{1}{2}$

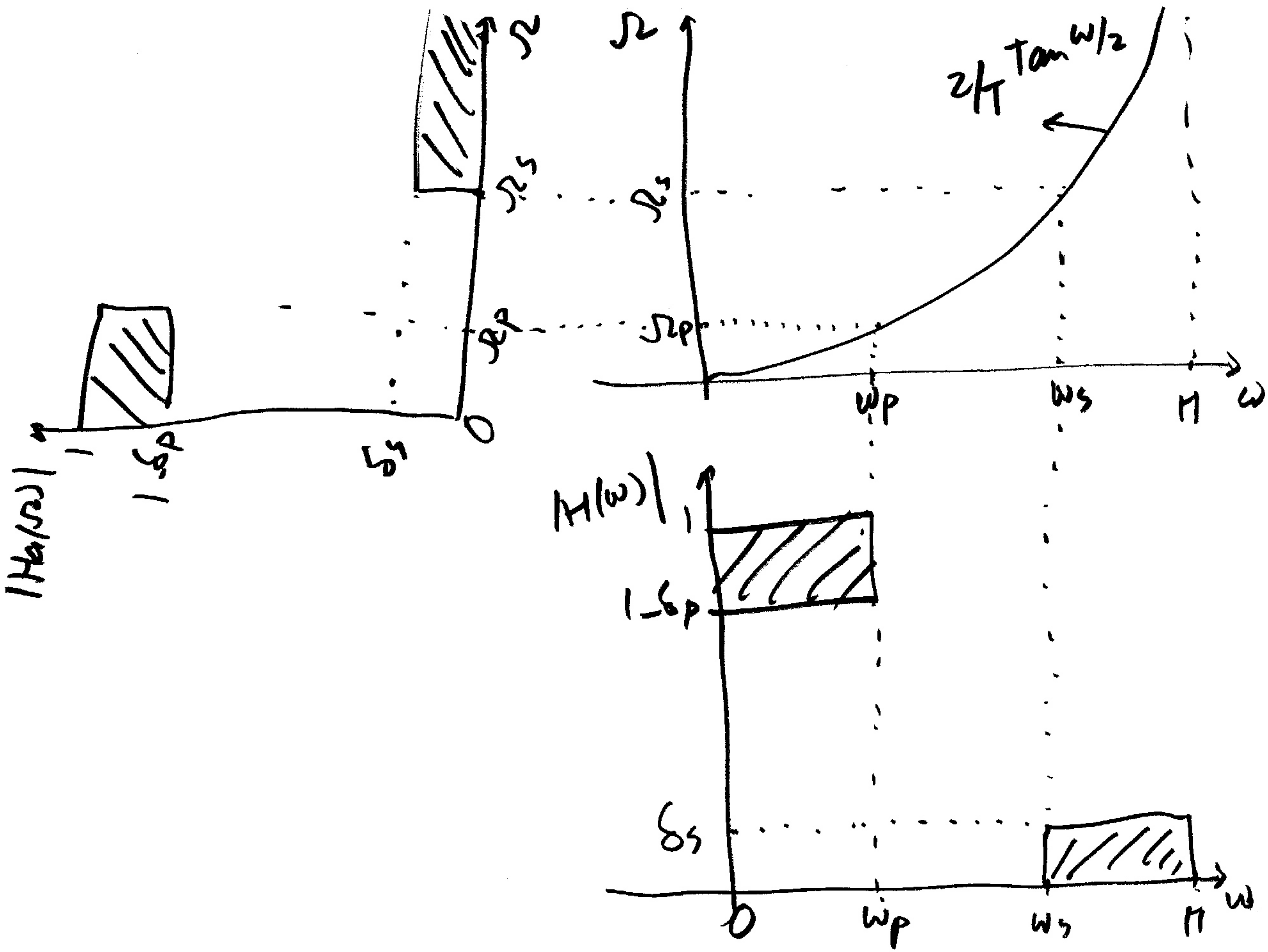
$\Rightarrow$ LHP $\longrightarrow$ inside unit circle.

$\Rightarrow$ causal stable $H_a(s) \longrightarrow$ causal stable $H(z)$



Lets use this to translate specs

D.T $\longrightarrow$ Analog.


Ω
Ω_s
Ω_p
0
δ_s
$1-\delta_p$
1
$|H_a(\Omega)|$
$\frac{2}{T}\tan\omega/2$
ω_p
ω_s
π
ω
$|H(\omega)|$