

Dec. 5, 2006

# Discrete Fourier Series

real.

$$x(t) \xrightarrow{\text{C.T.F.T.}} X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

C.T.F.T.  
C.T.F.F.T.

continuous

int.

$$x(n) \xrightarrow{\text{D.T.F.T.}} X(\omega) = \sum_n x(n) e^{-j\omega n}$$

D.T.F.T.  
real.

discrete time signal

$$x(n) \xrightarrow{\text{Z.T.}} X(z) = \sum_n x(n) z^{-n}$$

Z.T.  
complex.

discrete time

$$x(n) \xrightarrow{\text{D.F. Series.}} X(k) = \sum_n x(n) e^{-j2\pi nk/N}$$

D.F. Series.

periodic

$$x(n) \xrightarrow{\text{D.F.T.}} X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

D.F.T.

finite length

finite length

# DFS - Discrete Fourier Series

Deal with  $\tilde{x}(n)$  periodic, discrete time signal.

$$\tilde{x}(n) = \tilde{x}(n + kN) \leftarrow \text{any integer} = \text{period.}$$

Idea: Decompose  $\tilde{x}(n)$  in terms of exponentials.  
periodic with period  $N$ .

$$e^{j\frac{2\pi nk}{N}} \leftarrow \text{periodic with period } N, k=0, \dots, N-1$$

There are  $N$ , periodic exponentials with period  $N$ .

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j\frac{2\pi nk}{N}} \leftarrow \text{weights}$$

$e_k(n)$  is periodic with period  $N$ :

$$e_k(n) \stackrel{??}{=} e_{k+rN}(n)$$

arbitrary int.

$$e^{j2\pi nk/N}$$

$$e^{j2\pi n(k+rN)/N}$$

Proof:

$?$

$$e^{j2\pi nr \frac{rN}{N}}$$

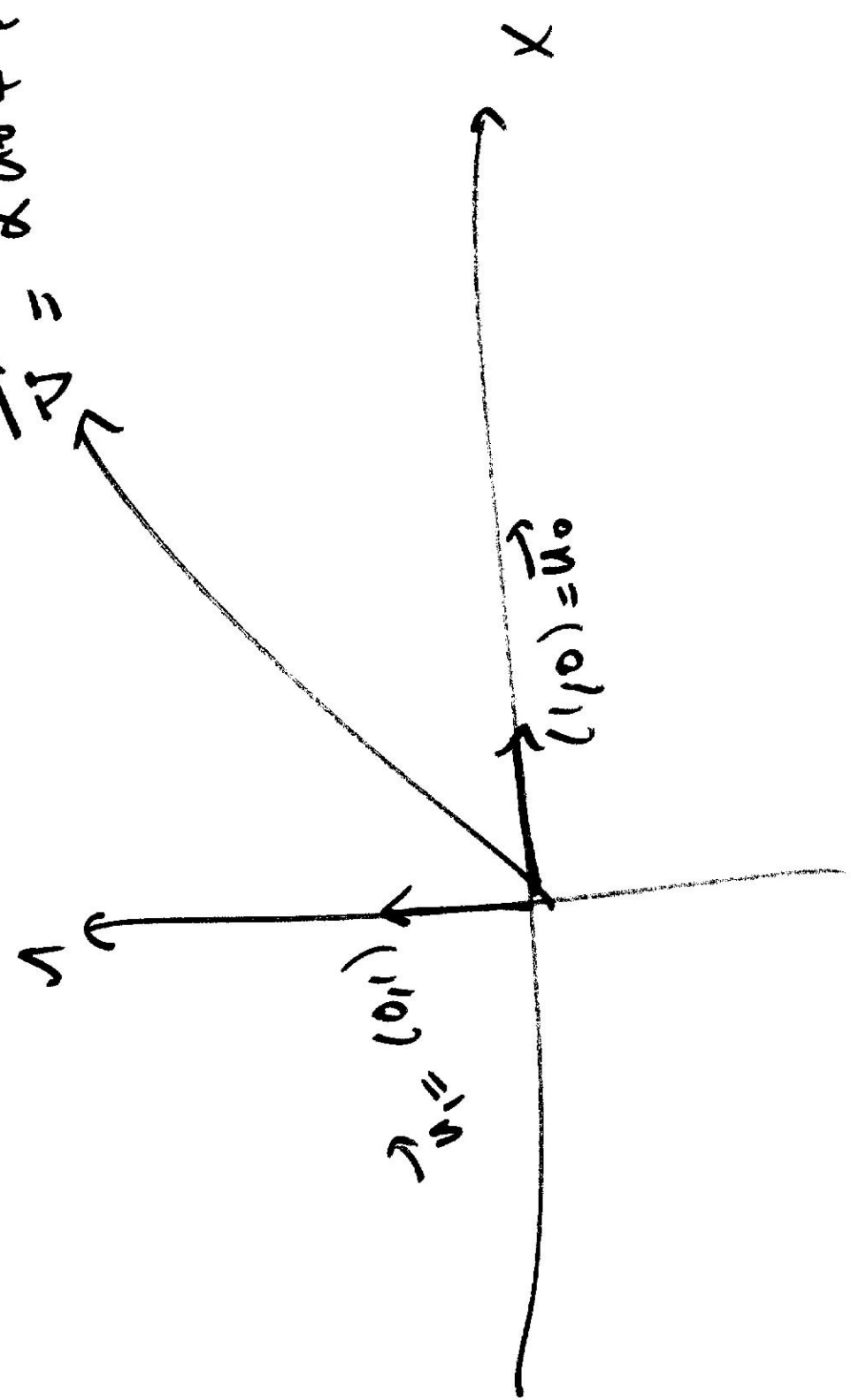
$$e^{j2\pi kr/N}$$

$$e^{j2\pi nk/N}$$

$$e_0(n) = e_N(n) = e_{2N}(n) = e_{3N}(n) = \dots$$

$$e_1(n) = e_{N+1}(n) = e_{2N+1}(n) = \dots$$

$$\vec{u} = \alpha \vec{u}_0 + \beta \vec{u}_1$$



Q How find "weight"?  $X(k)$ ?

proposal: 
$$X(k) \stackrel{??}{=} \sum_{n=0}^{N-1} \underbrace{X(n)} e^{-j2\pi nk/N}$$

proof: 
$$X(k) \stackrel{??}{=} \sum_{n=0}^{N-1} \left( \frac{1}{N} \sum_{l=0}^{N-1} X(l) e^{j2\pi(l-k)n/N} \right) e^{-j2\pi nk/N}$$

(A)

what is (A)?

$$\delta(l-k-rN) = X(k+rN) \stackrel{\text{int.}}{\rightarrow \text{obs.}} \delta$$

$$\sum_{l=0}^{N-1} X(l) \delta(l-k-rN) = X(k+rN)$$

Case 1: If  $l-k$  is an int. multiple of  $N$ .

$$\textcircled{A} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j 2\pi r N n} = 1$$

if  $k$  is not an int. multiple of  $N$ .

Case 2  $l-k \neq rN$   $\sum_{n=0}^{N-1} e^{j 2\pi (l-k)n / N}$

$$\textcircled{A} = \frac{1}{N} \sum_{n=0}^{N-1} e^{j 2\pi \alpha n} = \frac{1 - \alpha^N}{1 - \alpha}$$

Recall  $\sum_{n=0}^{N-1} \alpha^n = \frac{1 - \alpha^N}{1 - \alpha}$

$$\textcircled{A} = \frac{1}{N} \frac{1 - e^{j 2\pi (l-k)N / N}}{1 - e^{j 2\pi (l-k) / N}} = 0$$

$$\textcircled{A} = \delta(l-k - rN)$$

$$\bar{X}(k) = X(k + rN) \quad r = \text{arb. int.}$$

$\Rightarrow X(k)$  is a periodic sequence with period  $N$ .

$\Rightarrow$  From now on refer to  $X(k)$  as

$$X(k)$$

DFS pair.

$$\checkmark \tilde{X}(k) = \sum_{n=0}^{N-1} \checkmark \tilde{x}(n) e^{-j \frac{2\pi n k}{N}}$$

Analysis.

$$\checkmark \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \checkmark \tilde{X}(k) e^{+j \frac{2\pi n k}{N}}$$

Synthesis

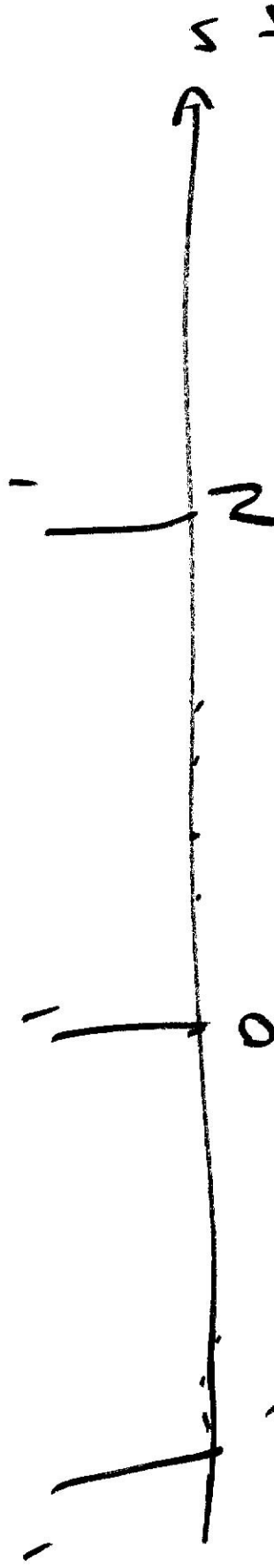
DFS:

periodic  
N pt seq  
 $\checkmark \tilde{x}(n)$

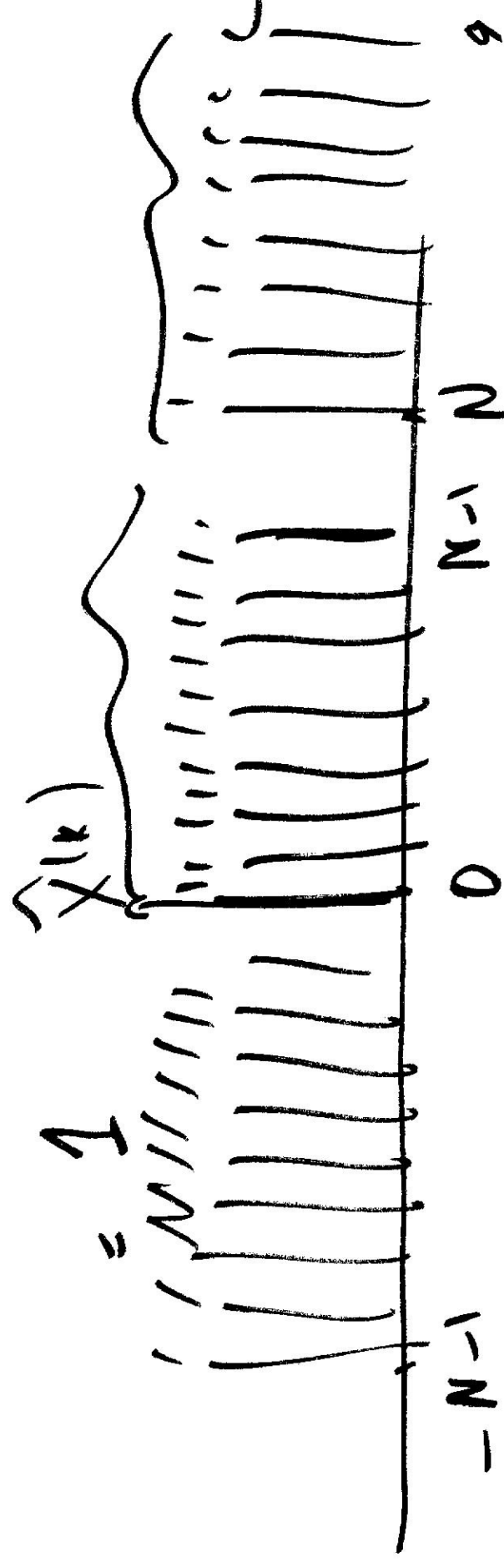
periodic  
N point  
seq is freq.  
periodic  
 $\checkmark \tilde{X}(k)$



$$\underline{\text{Ex}} \quad \hat{x}(n) = \sum_{r=-A}^B \delta(n+rN)$$



$$\hat{X}(k) = \sum_{n=0}^{N-1} \left( \sum_{r=-A}^{+B} \delta(n+rN) \right) e^{-j2\pi nk/N}$$



equation  
train.

$$e^{j2\pi nk/P}$$

$$\sum_{k=0}^{N-1}$$

$$\frac{1}{N}$$

$$\sum_{r=-\infty}^{+\infty} \delta(n+rN)$$

$$r=-\infty$$

$$\hat{x}(n)$$

$$\hat{x}(n)$$

# Shift Property

$$\begin{aligned}
 & \checkmark x(n) \longleftrightarrow \checkmark X(k) \\
 & \checkmark x(n-n_0) \longleftrightarrow \checkmark e^{-j2\pi n_0 k} X(k)
 \end{aligned}$$

# Periodic Convolution

$\checkmark$  period  $N$ .

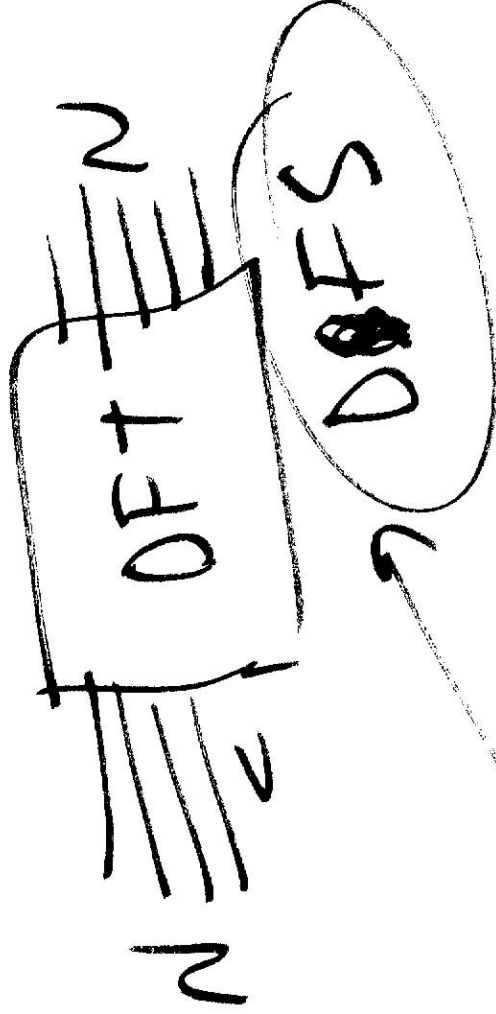
$$\begin{aligned}
 & \checkmark x_1 \quad \checkmark x_2 \quad \checkmark x_3 \\
 & \text{Period } N \quad \text{Period } N \quad \text{Period } N \\
 & \checkmark x_1 * \checkmark x_2 = \checkmark x_3 \\
 & \checkmark \sum_{m=0}^{N-1} \checkmark x_1(m) \checkmark x_2(n-m)
 \end{aligned}$$

$$\checkmark X_3(k) = \checkmark X_1(k) \checkmark X_2(k)$$

DFT = Discrete Fourier Transform.

$x(n)$

$X(k)$  NPT seq



DFT

DTFT.

# First Approach To DFT via DFS

1. Start with a finite extant seq  $x(n)$

$N$  points long  $n=0, \dots, N-1$  with  $\tilde{x}(n)$

2. "Periodize"  $x(n)$  to get  $\tilde{x}(n) R_p(n)$  with  $\tilde{x}(n)$  extra one period of  $x(n)$

$$x(n) = \begin{cases} \tilde{x}(n) & n=0, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$R_p(n) = \begin{cases} 1 & n=0, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{x}(n) = \sum_{k=-\infty}^{+\infty} x(n + rN) \leftarrow \text{periodicization}$$

3. Take DFS of  $\tilde{x}(n) \rightarrow \tilde{X}(k)$
4. Take one period of  $\tilde{X}(k)$  to get

$$\tilde{X}(k) = \text{DFT of } x(n)$$

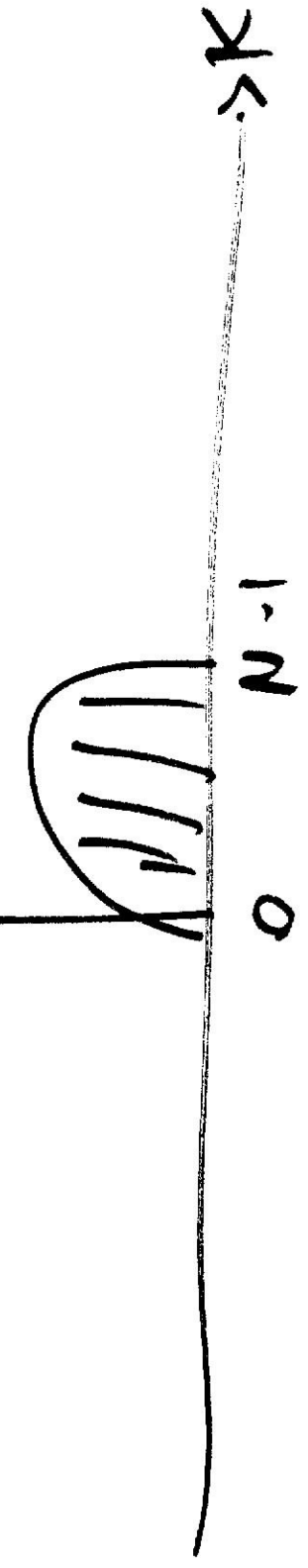
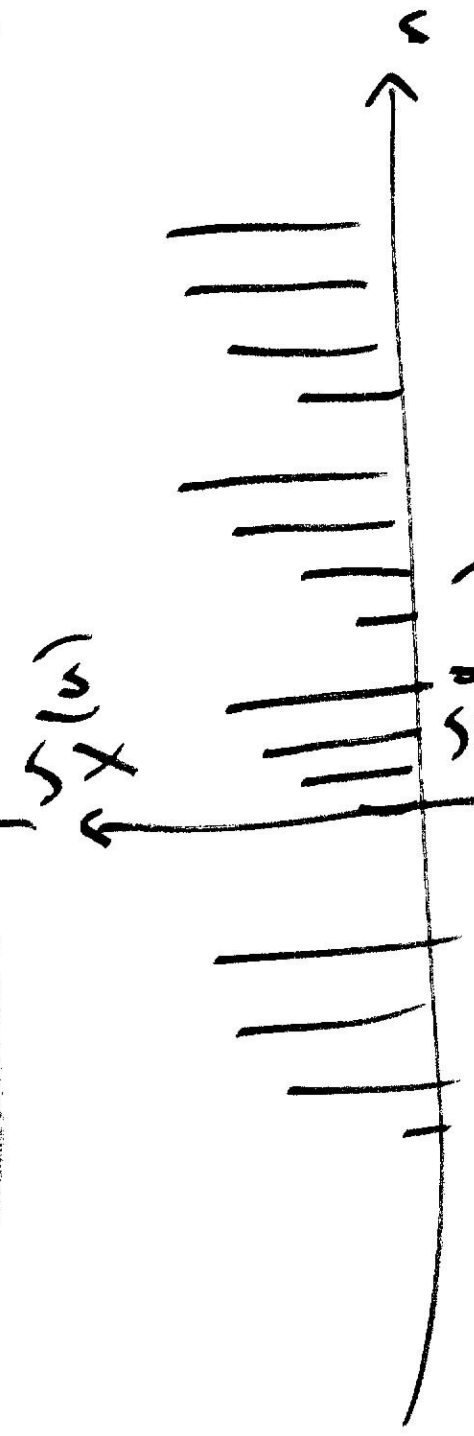
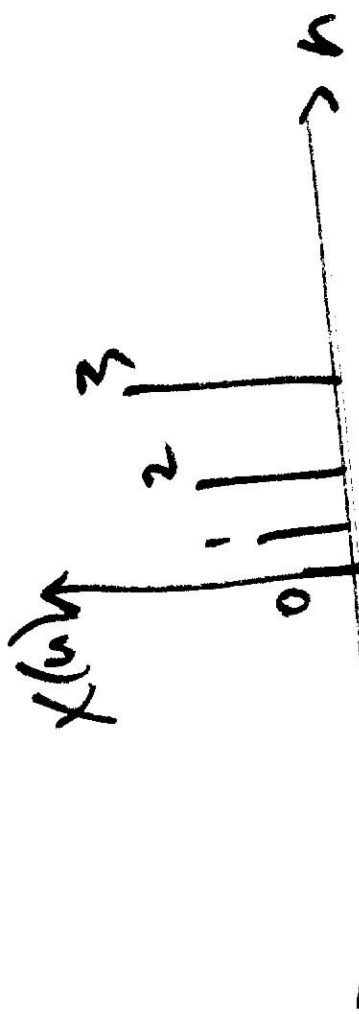
$$\tilde{X}(k) = \tilde{X}(k) R_N(k)$$

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$$x(n) \xrightarrow{\text{DFS}} \tilde{X}(k) \xrightarrow{\text{DFS}} \tilde{X}(k)$$

$\tilde{X}(k)$  is periodic with  $N$  pts.  
 $\tilde{X}(k)$  is periodic with  $N$  pts.

Ex 4



# Defn of DFT

$$X(k) = N_p + \text{DFT of } x(n) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad 0 \leq k < N$$

0

otherwise

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}$$

0

otherwise

$0 \leq n < N$



Relate DFT to DTFT:

$$\sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

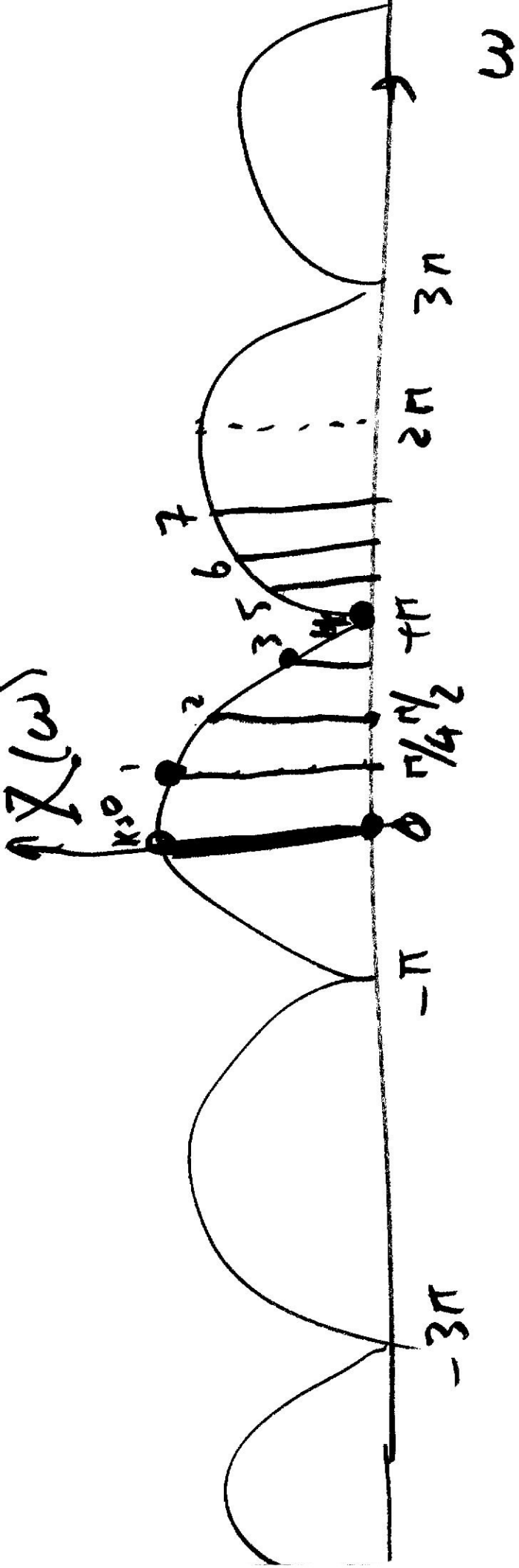
$$0 \leq k < N$$

$$X(k) = \begin{cases} [X(\omega)]_{\omega = \frac{2\pi k}{N}} \\ 0 \end{cases}$$

otherwise.

DFT is equally spaced samples of

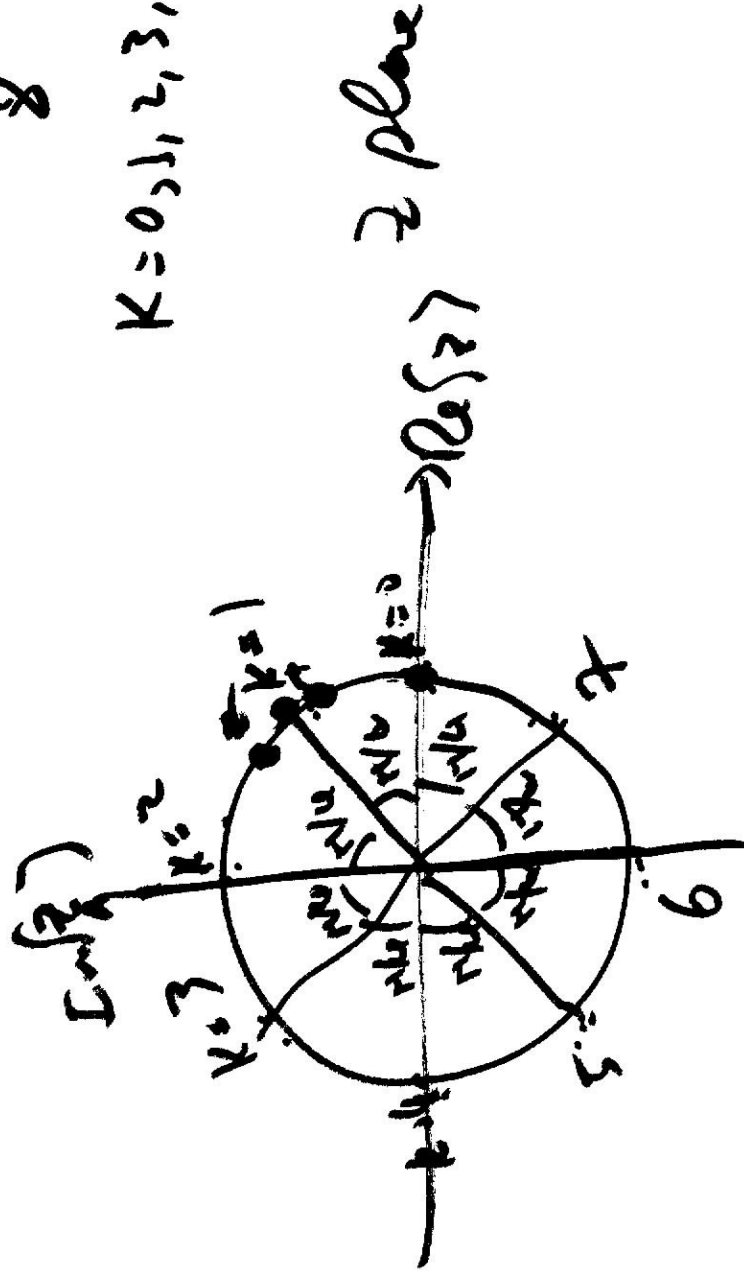
DTFT.



80 pt 507.  $\rightarrow$  4PT DFT.

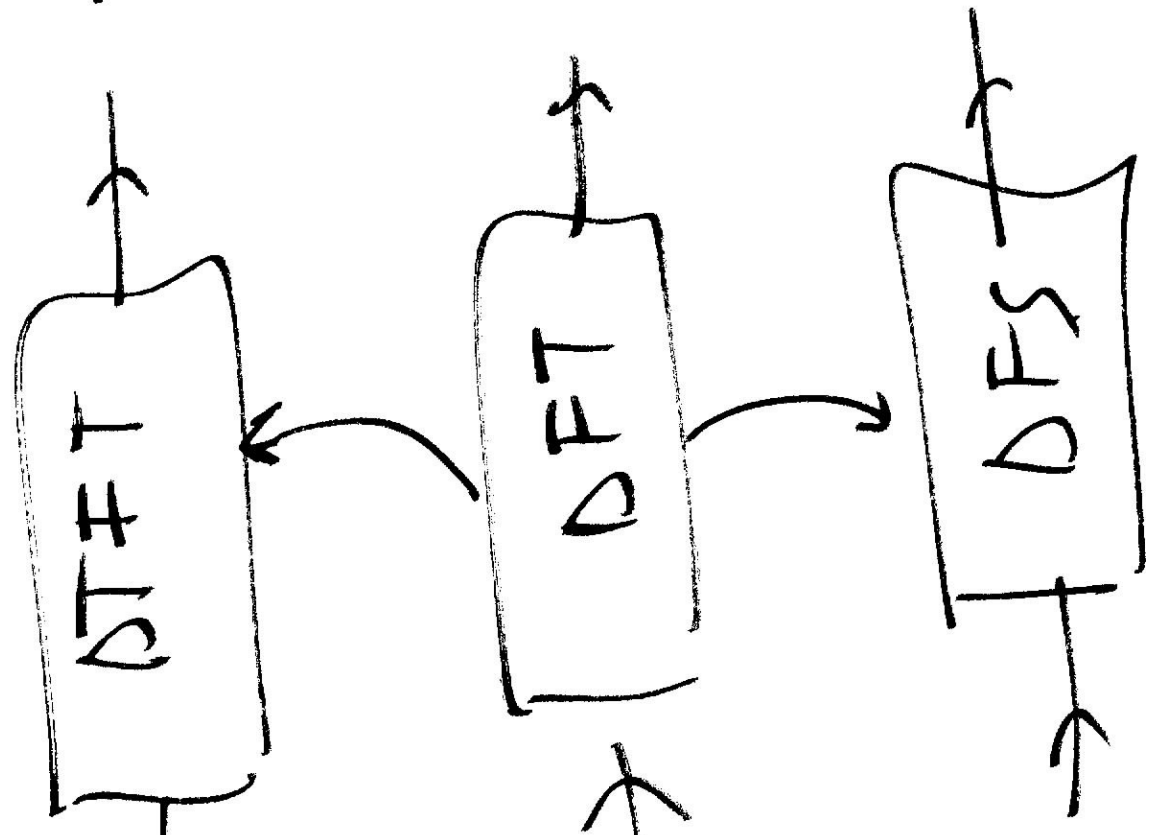
$$\omega = \frac{2\pi k}{8}$$

$$k = 0, 1, 2, 3, 4, 5, 6, 7$$



$X(\omega)$   
Real.

$X(k)$   
int  
 $X(k)$



$x(n)$

integers

$x(n)$

int.

finite

~~ext~~ extent

$x(n)$

periodic