Convergence Issues

$$X(\omega) = \text{D.T.F.T.}\ \{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Q: Does it converge?

If yes, what does it converge to?

$\longrightarrow Y(\omega)$.

Q If it converged to $Y(\omega)$, plug in $Y(\omega)$ into Inverse F.T. formula do I get $x(n)$ back?

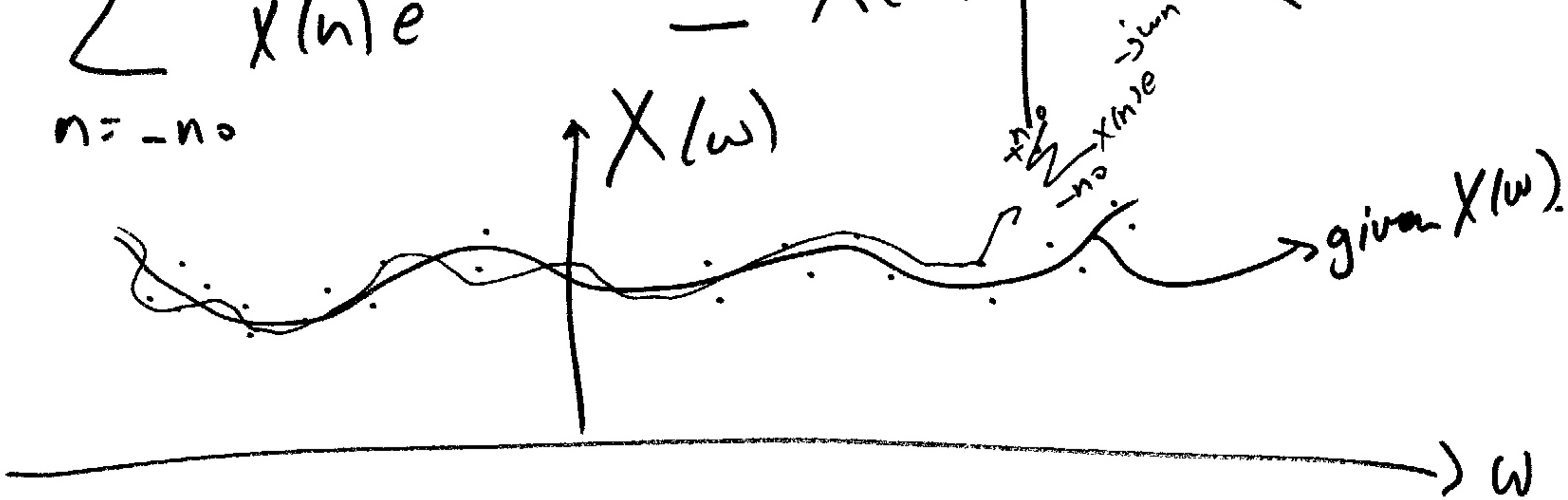
$$\int Y(\omega)\, e^{j\omega n}\, d\omega \overset{??}{=} x(n)$$

Def <u>uniform</u> <u>Convergence</u>. $\sum_n x(n) e^{-j\omega n}$

converges uniformly if $\exists$ a continuous function of ω, called $X(\omega)$ such that.

$\boxed{\forall\ \epsilon, \omega}$, $\exists\ n_0$ such that.
$\epsilon > 0$

$$\left| \sum_{n=-n_0}^{+n_0} x(n) e^{-j\omega n} - X(\omega) \right| < \epsilon$$

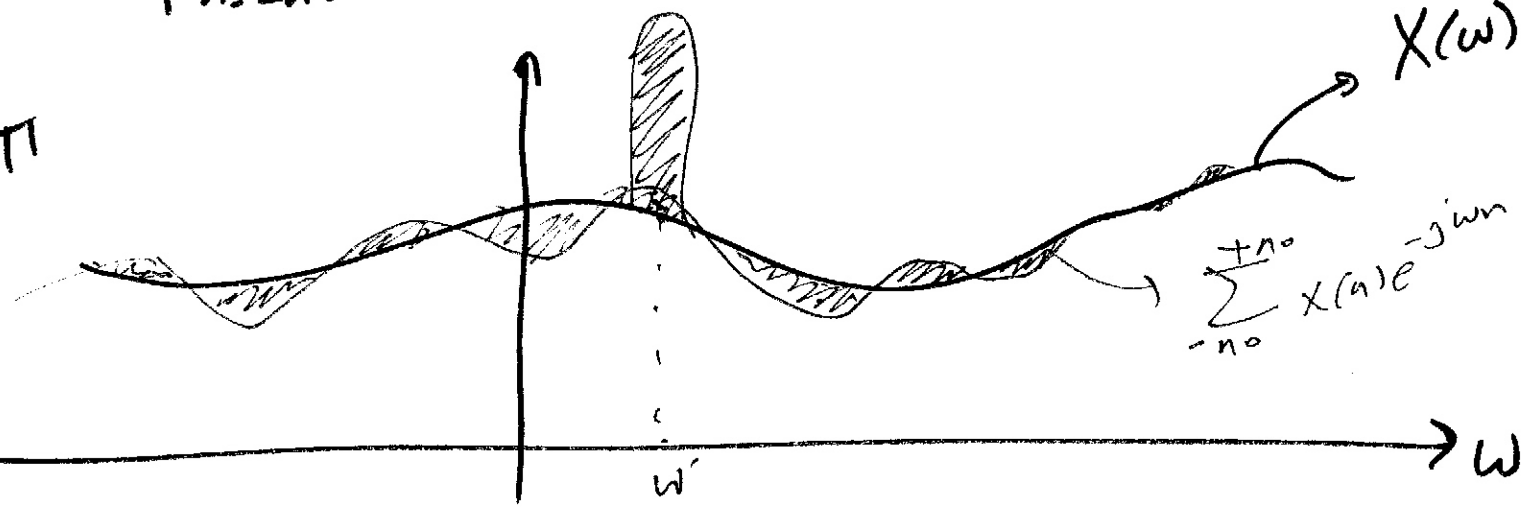


mean Square Error Convergence (MSE).

$\sum_n x(n) e^{-j\omega n}$ converges in mse sense to a known, given fn, $X(\omega)$ if

$\forall \epsilon$, $\exists n_0$ s.th.

$$\int_{-\pi}^{+\pi} \left| \sum_{n=-n_0}^{+n_0} x(n) e^{-j\omega n} - X(\omega) \right|^2 d\omega < \epsilon$$



① If seq $x(n)$ is abs sumable,

$\sum_n |x(n)| < \infty$ Then $\sum_n x(n) e^{-j\omega n}$

converges uniformly $\Rightarrow$ D.T.F.T. exists.

and DTFT is a continous function.

Furthermore, whatever $\sum_n x(n) e^{-j\omega n}$ converges

to, if I plug into (call it $X(\omega)$), $\int X(\omega) e^{j\omega n} d\omega$

I get back $x(n)$.

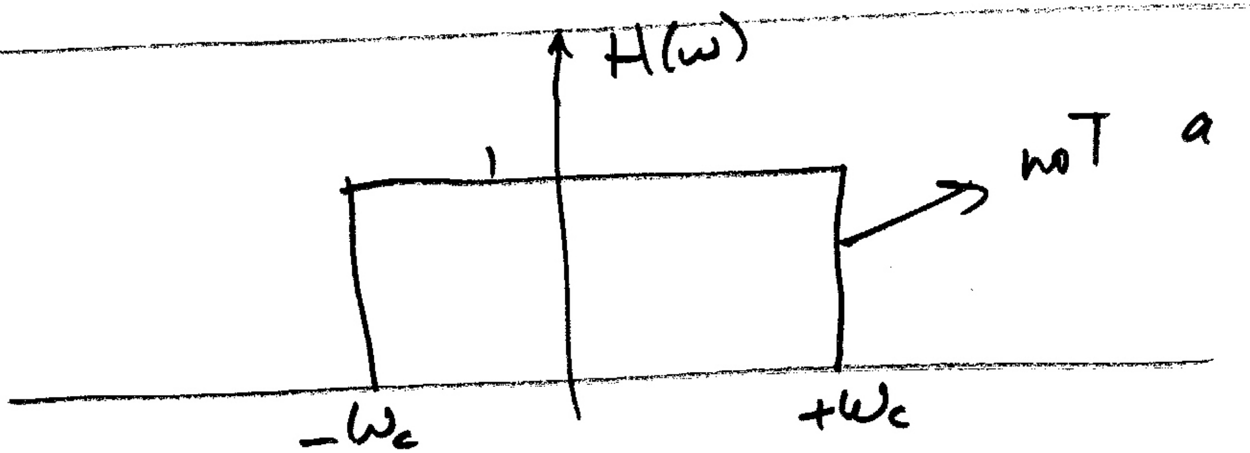
<u>Ex</u>: $\left(\frac{1}{2}\right)^n u(n) = x(n) \longrightarrow \dfrac{1}{1 - \frac{1}{2} e^{-j\omega}}$

$2^n u(n) = x(n) \longrightarrow$ F.T. doesn't exist.

(2) If $x(n)$ is square summable,

$$\sum_n x^2(n) < \infty \Longrightarrow \text{convergence in mse sense.}$$

Ex



$$\int_{-\pi}^{+\pi} H(\omega)\, e^{j\omega n}\, d\omega = \frac{\sin w_c n}{\pi n} = h(n).$$

$h(n)$ is <u>not</u> abs. summable

$\sum_n \frac{1}{n} \longrightarrow$ diverges ; so

Is $h(n)$ square summable $\sum_n \frac{\sin^2 w_c n}{\pi^2 n^2}$? $\sum_n \frac{1}{n^2}$?

converge

Since $\sum_n \frac{1}{n^2}$ converges

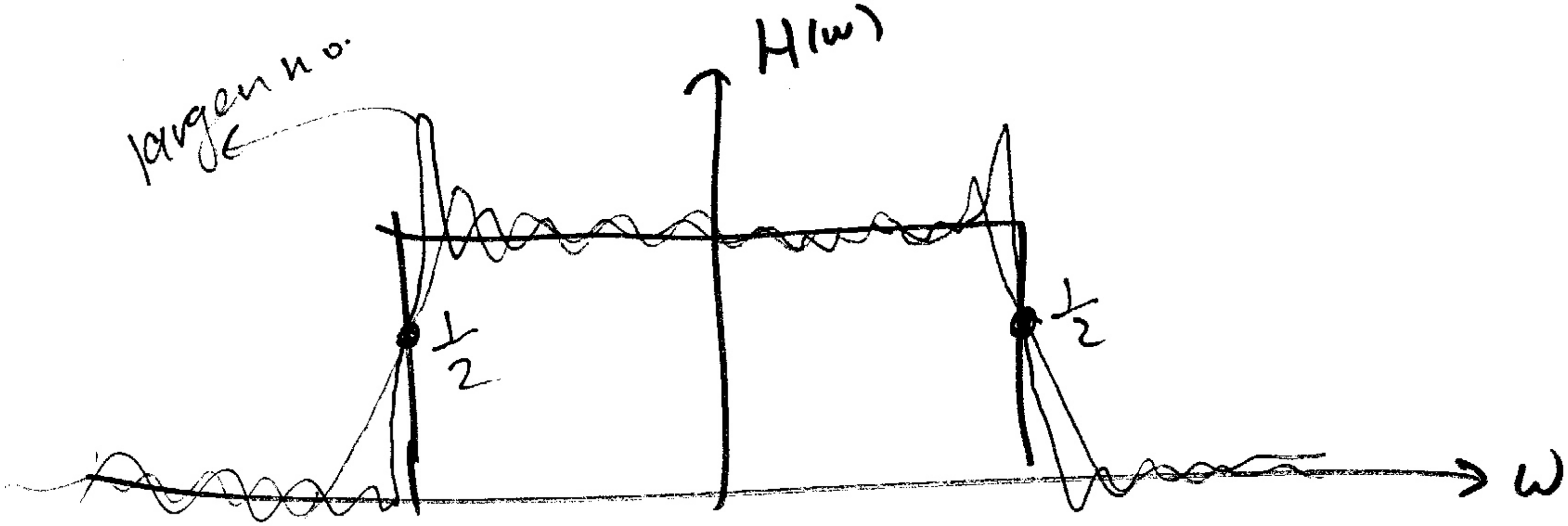
$\Longrightarrow h(n)$ is square summable $\Longrightarrow$

$\sum h(n) e^{-j\omega n}$ converges in a

mse sense.

Can show: $\sum_{-n_0}^{+n_0} h(n) e^{-j\omega_c n} = \frac{1}{2}$

at ω_c $\quad \forall n_0$.



$$\sum_{n_0} h(n) e^{-j\omega n}$$

IIR Filters with Rational Transfer fn. can be implemented with D.E.

Ex $h(n) = a^n u(n) \longleftrightarrow H(z) = \dfrac{1}{1 - a z^{-1}}$

$$x \longrightarrow \boxed{\begin{matrix} h(n) \\ H(z) \end{matrix}} \longrightarrow y(n)$$

$$Y(z) = X(z)\, H(z)$$

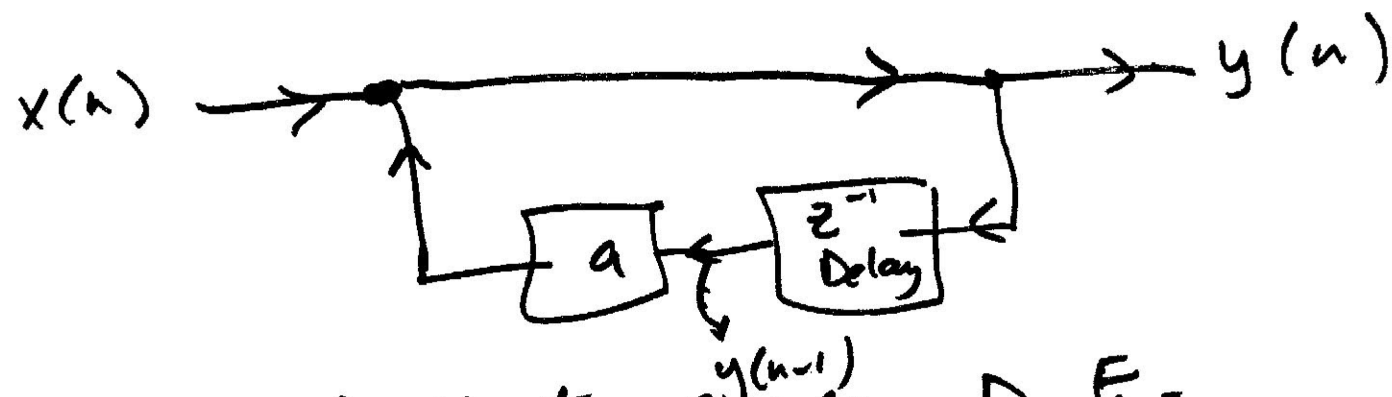
$$Y(z) = X(z) \frac{1}{1 - a z^{-1}}$$

$$Y(z)\left[1 - a z^{-1}\right] = X(z)$$

$$Y(z) - a z^{-1} Y(z) = X(z)$$

(Inverse Z.Transform

$$\boxed{y(n) - a\,y(n-1) = x(n)}$$ ← D.E.



Linear Constant Coeff D.E =

L C C DE.

Linear: no $x^2(n)$ no $x^2(n-1)$ no $x(n)\,y(n)$

Constant coeff: a is constant: Doesn't change with Time not $a(n)$

$y(n) - a(n)\,y(n-1) = x^2(n)$ → not linear

not a const coeff

Q: Under what conditions does L.C.C. D.E. correspond to an LTI system? or a causal LTI system?

EX $y(n) = a\,y(n-1) + b\,x(n)$

claim Is not even a system.

system: unique input $\longrightarrow$ unique output.

proof: Suppose $y_1(n)$ is a soln.

Then: $y_2(n) = y_1(n) + K a^n$ is also a soln.

if $y_1(n)$ soln $\longrightarrow$ $y_1(n) = a\,y_1(n-1) + b\,x(n)$

Show y_2 is also a soln:

$$y_2(n) \stackrel{?}{=} a\,y_2(n-1) + b\,x(n)$$

$$y_1(n) + Ka^n \stackrel{?}{=} a\left[y_1(n-1) + Ka^{n-1}\right] + b\,x(n)$$

✓ True. ie y_2 is also a soln.

<u>Fix</u> Add I.C. = Initial Condition.

<u>Claim</u> with I.C. $\longrightarrow$ system.

$$y(n) = a\,y(n-1) + x(n)$$

$$x(n) = b\,\delta(n)$$

I.C. $\quad y(-1) = y_0$

<u>claim</u>: Can find a unique output.

① by inspection: $\quad n \geq 0$:

$$y(0) = a\,y(-1) + b\,\delta(0)$$
$$= a y_0 + b$$
$$y(1) = a\,y(0) + b\,\delta(1)$$
$$= a(a y_0 + b)$$
$$= a^2 y_0 + ab$$

$n \geq 0 \qquad \boxed{y(n) = a^n (a y_0 + b)}$

$n \leq -2$ $\quad y(n) = a\, y(n-1) + x(n)$

$$a\, y(n-1) = \frac{1}{a}\, y(n) - \frac{b}{a}\, \delta(n)$$

$n = -1$

$$y(-2) = \frac{1}{a}\, y(-1) - \frac{b}{a}\, \delta(-1)$$

$$= \frac{1}{a} y_0 - \frac{b}{a}\, \delta(-1)$$

$$= \frac{1}{a} y_0$$

$$y(-3) = \frac{1}{a^2} y_0$$

$$\vdots$$

$$y(n) = a^{n+1} y_0$$

$n \leq -1$ $\quad \boxed{y(n) = a^{n+1} y_0}$

$$y(n) = a^n (a y_0 + b)\, u(n) + u(-n-1)\, a^{n+1} y_0$$

$\forall n \quad \boxed{y(n) = a^{n+1} y_0 + a^n b \, u(n)}$

Aliter:

(a) find homogeneous soln $y_h(n)$

(b) find particular soln $y_p(n)$

(c) $y(n) = y_h(n) + y_p(n)$

(d) Impose I.C.

(a) homogeneous soln: output if input is set to zero.

$$y(n) = a\, y(n-1)$$

guess: $y_h(n) = K \alpha^n$

$$K\alpha^n = a K \alpha^{n-1} \Longrightarrow \boxed{y_h(n) = K a^n}$$

(b) particular soln:

$$y(n) = a\,y(n-1) + b\,\delta(n)$$

$$Y(z) = a z^{-1} Y(z) + b$$

$$Y(z) = \frac{b}{1 - a z^{-1}}$$

$|z| > |a|$ b.1

$|z| < |a|$ b.2

b.1 $$y_{p_1}(n) = b\,a^n u(n)$$

b.2 $$y_{p_2}(n) = -b\,a^n u(-n-1)$$

(c) combine particular soln with homogen.

$$y_{Tot} = y_h(n) + y_p(n)$$

(d) Impose I.C. $\to$ d.1, d.2

d.1 $\quad y_{Total}(n) = K a^{n} + b a^{n} u(n)$

Apply I.C. $n=-1$ $\quad y_{Tot}(-1) = K a^{-1} + b a^{-1} u(-1) = y_0$

$$\Rightarrow K = a y_0$$

d.1 $\quad \boxed{y_{Tot}(n) = y_0 a^{n+1} + b a^{n} u(n)}$

d.2 $\quad y_{Tot}(n) = K a^{n} - b a^{n} u(-n-1)$

Apply I.C. $\quad y_{Tot}(-1) = y_0 = K a^{-1} - b a^{-1} u(0)$

$$\Rightarrow K = a y_0 + b$$

$$y_{Tot}(n) = a^n (a y_0 + b) - b a^n u(-n-1)$$

d.2

$$\boxed{y_{ToT}(n) = a^{n+1} y_0 + b a^n u(n)}$$

$\longrightarrow$ got same answer

Not a linear system.

zero input ie $b=0$ doesn't result in zero output.

$b=0 \longrightarrow y_{Tot}(n) = a^{n+1} y_0 \longrightarrow$ not zero.

If $y_0 = 0 \longrightarrow$ Does correspond to a linear syst.

1. L.C.C.DE. needs I.C. to become a system
2. L.C.C.D.E needs Zero I.C. To become a Linear system.
3. Q: Does this LCCDE correspond To T.I. system?

Let's shift input $w(n) = b\,\delta(n-n_0)$

output due To $w(n)$: $a^{n+1} y_0 + b a^{n-n_0} u(n-n_0)$

old output shifted by n_0 : $a^{(n-n_0)+1} y_0 + b\,a^{n-n_0} u(n-n_0)$

Can make it T.I. if $y_0 = 0$

~~Let y_0~~ If let $y_0 = 0 \Longrightarrow$ T.I.

(3) LCCDE + zero I.C. $\longrightarrow$ Linear + T.I.

Claim: If you also want causality $\Longrightarrow$ Initial Rest Condition. IRC

LCCDE + I.R.C. $\longleftrightarrow$ LTI + causality

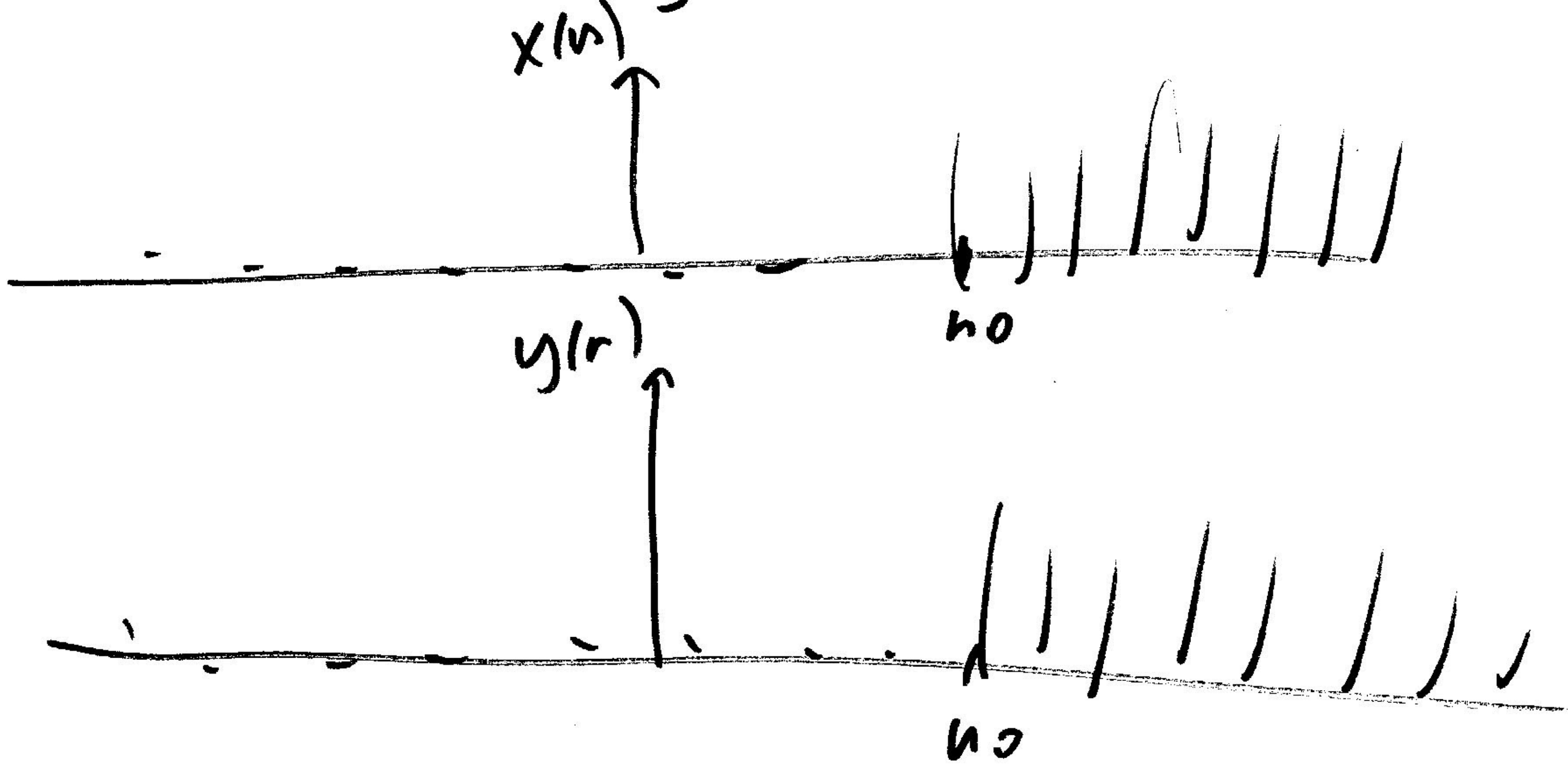
__I.R.C__ sys has I.R.

iff.

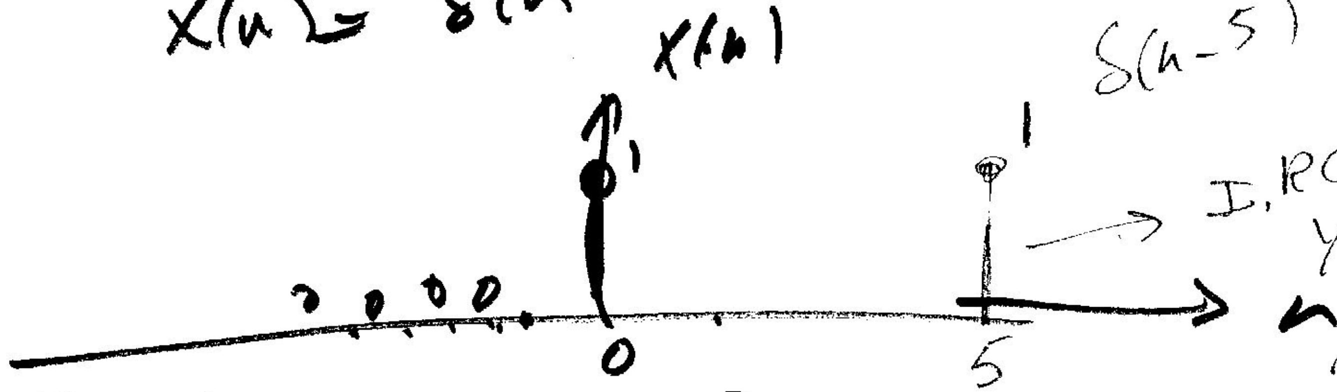
for $x(n) = 0$ $n < n_0$

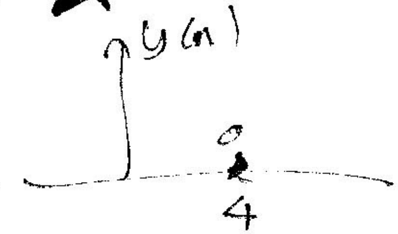
results in

$y(n) = 0$ $n < n_0$



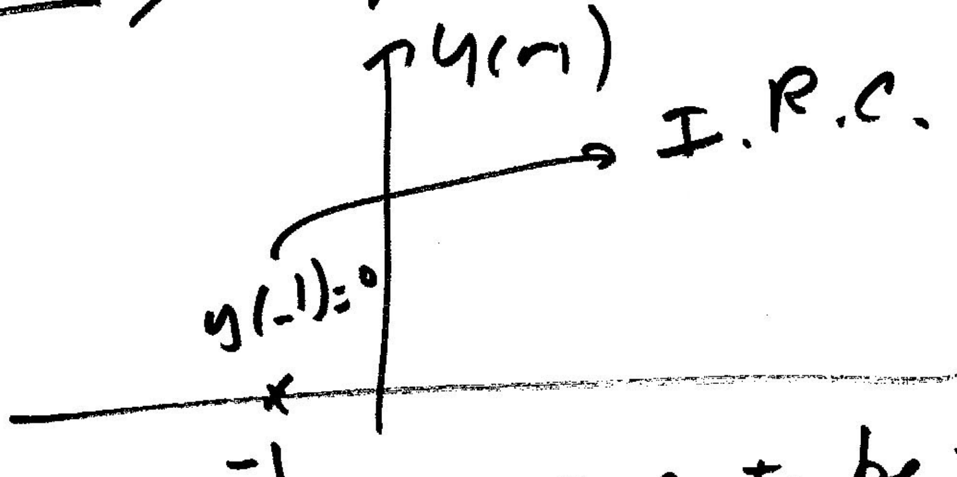
Ex $x(n) = \delta(n)$



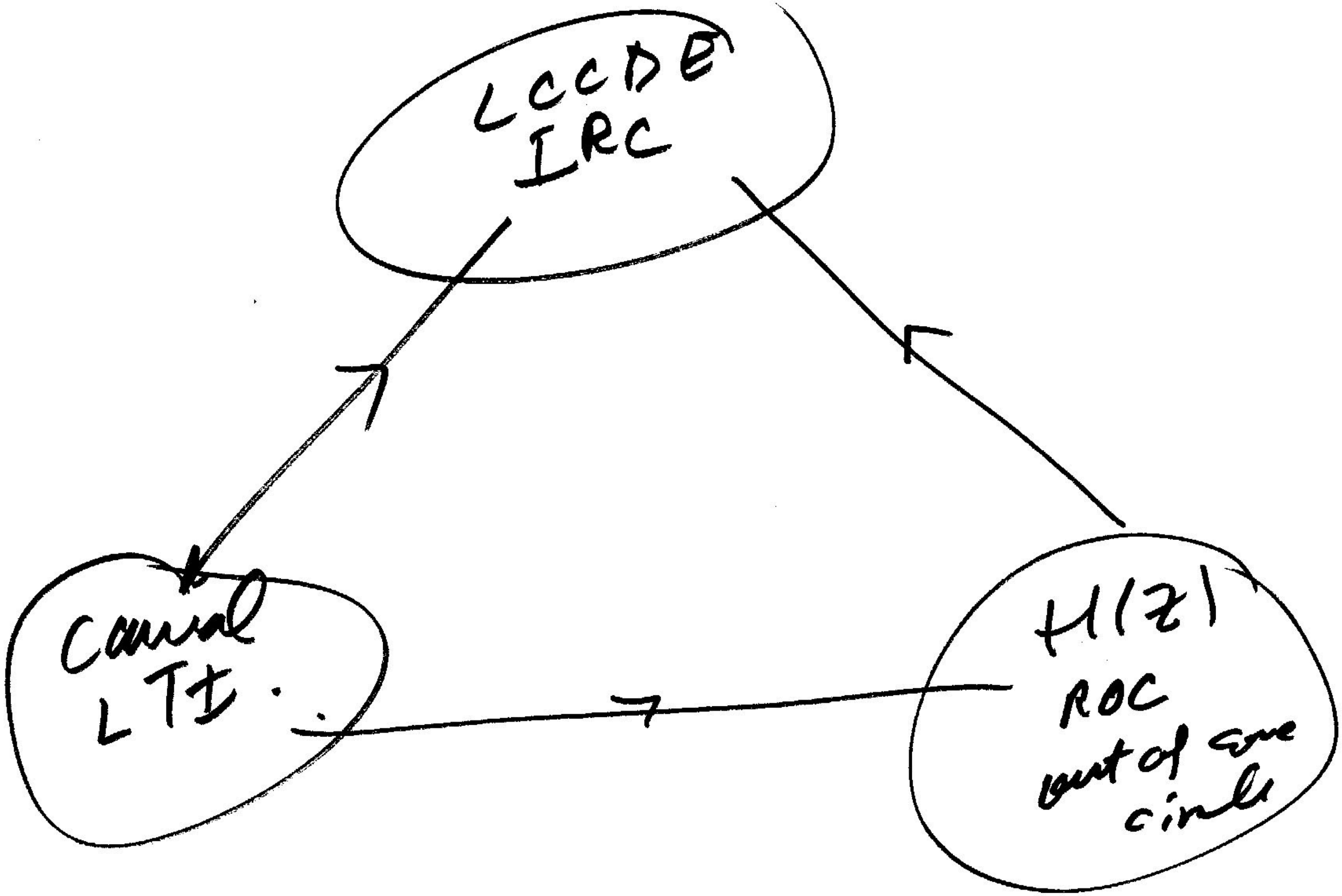


First order LCCDE.

I.RC. $\longrightarrow$ output $(-1) = 0$



I.RC. means you choose I.C to be zero at the Time (n) that depends on x(n)


LCCDE
IRC
Causal
LTI.
H(z)
ROC
out of one
circle

Final Rest Condition

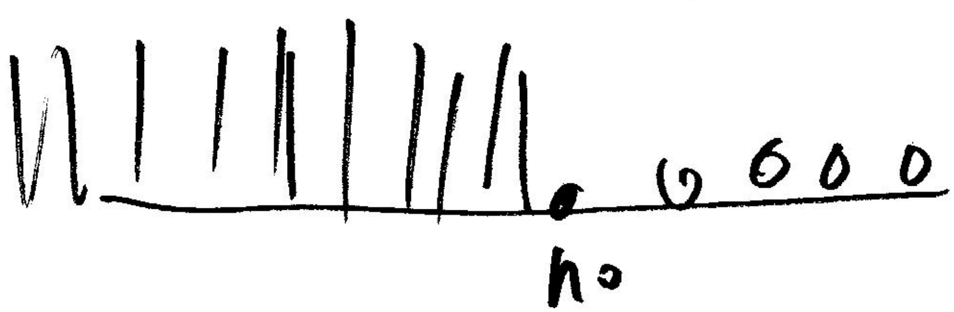
LCCDE + F.R.C. $\longleftrightarrow$ LTI + anti Causality

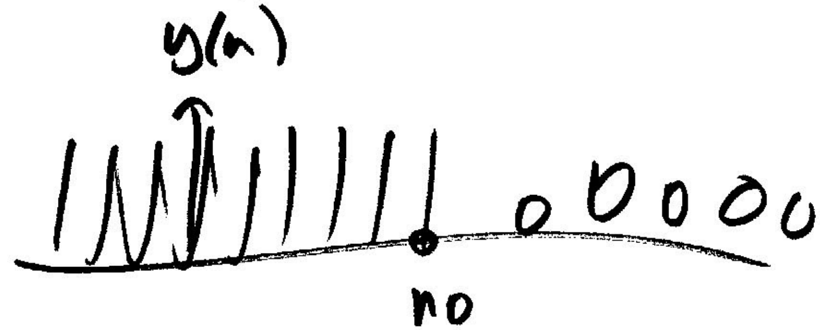
<u>F.R.C</u> sys has F.R.C. iff.

$x(n) = 0$ for $n > n_0$

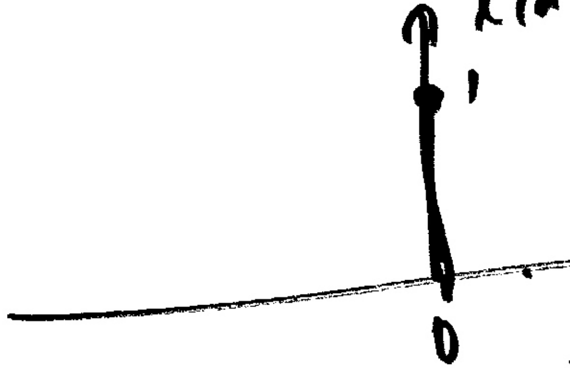
results in:

$y(n) = 0$ for $n > n_0$



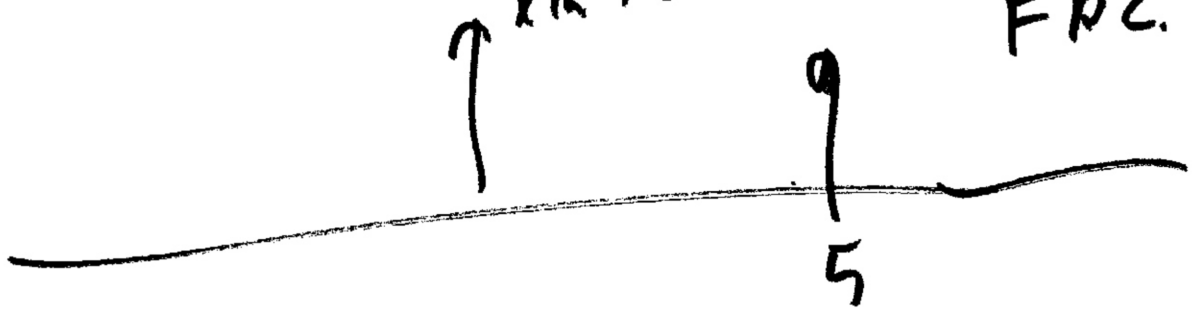


$x(n) = \delta(n)$



FRC $\rightarrow$ $y(0) = 0$

$x(n) = \delta(n-5)$



FRC $\quad y(+5) = 0$

$$y(n) = a\, y(n-1) + b\, \delta(n)$$

1. Causal : $\longrightarrow$ I.R.C. $\longrightarrow$ $y(-1) = 0$
2. Anticausal $\rightarrow$ F.R.C $\longrightarrow$ $y(\cancel{+}) = 0$

   $y(0) = 0$

↳ LCCDE could correspond to 2 different systems depending upon which I.C. I choose.

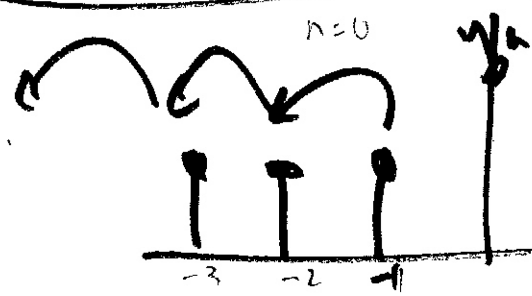
① Causal Realization

$$y(n) \leftarrow a\,y(n-1) + x(n)$$

computational procedure.

$x(n) = \delta(n)$

② Anticausal Realization:

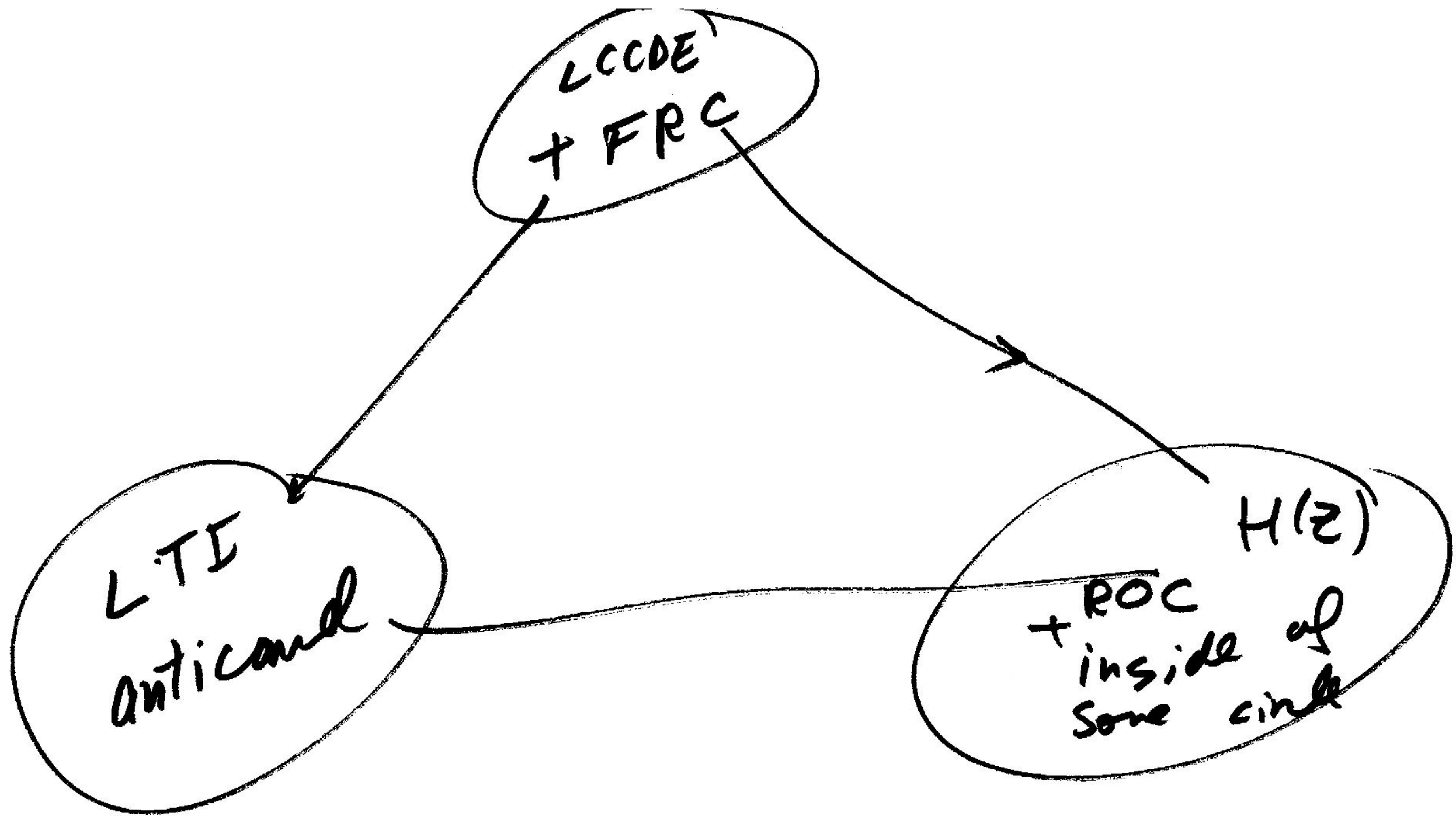


$$y(n-1) \leftarrow \frac{1}{a}y(n) - \frac{1}{a}x(n)$$

$$y(-1) \leftarrow \frac{1}{a}y(0) - \frac{1}{a}x(0)$$

$y(0)=0$

$$y(-2) \leftarrow \frac{1}{a}y(-1) + \frac{1}{a}x(-1)$$


LCCDE
+ FRC
LTI
anticausal
H(z)
+ ROC
inside of
some circle