Sept. 12, 2006

Different Realizations of L.C.C.D.E.

Recall a D.E corresponds to many different. computational Procedures.

Ex Consider a 2nd order ~~system~~ D.E:

$$y(n) = a\, y(n-1) + b\, y(n-2) + x(n)$$

3 Realizations of D.E.

Causal: ① $y(n) \leftarrow a\, y(n-1) + b\, y(n-2) + x(n)$

IRC: $y(-1) = y(-2) = 0$

Causal Implementation. System.
H(z) ROC outside some circle.

Anticausal: $y(-2) \leftarrow \frac{1}{b} y(0) - \frac{a}{b} y(-1) - \frac{1}{b} x(0)$

② $y(n-2) \leftarrow \frac{1}{b} y(n) - \frac{a}{b} y(n-1) - \frac{1}{b} x(n)$

FRC $y(0) = y(-1) = 0$ $\quad y(n) \leftarrow \frac{1}{b} y(n+2) - \frac{a}{b} y(n+1) - \frac{1}{b} x(n+2)$

$y(0) = y(-1) = 0$
if $x(n) = \delta(n)$

anticausal implementation, system.
$H(z)$ ROC inside some circle.

③ $y(n-1) \longleftarrow \frac{1}{a} y(n) - \frac{b}{a} y(n-2) - \frac{1}{a} x(n)$

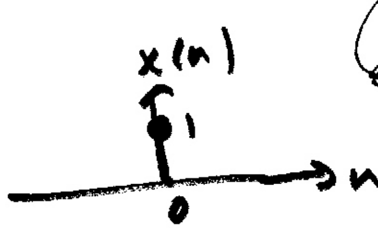
$y(n) \longleftarrow \frac{1}{a} y(n+1) - \frac{b}{a} y(n-1) - \frac{1}{a} x(n+1)$

non-causal.

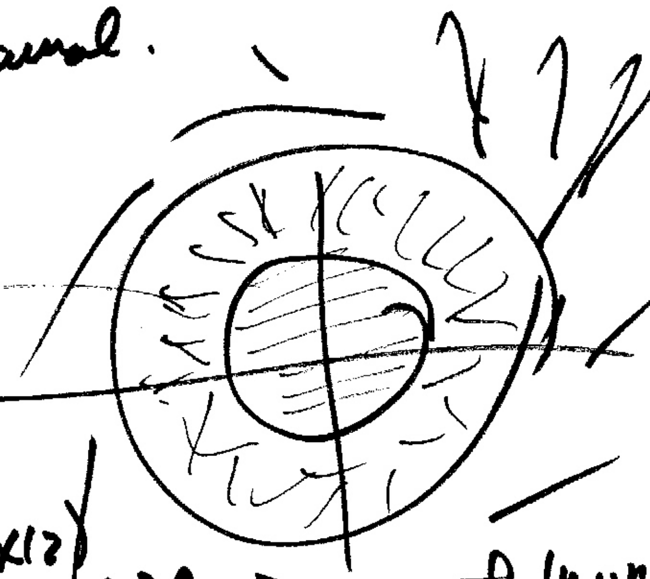
How about I.C:
$\left.\begin{array}{l} y(0) = 0 \\ y(-2) = 0 \end{array}\right\} \to y(-1)$

$x(n)$

$x(n) = \delta(n)$
what I.C.?



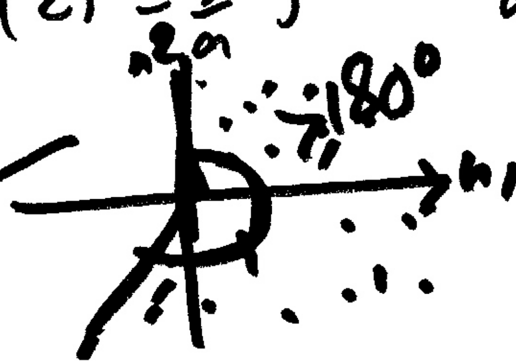
H(z) is ring



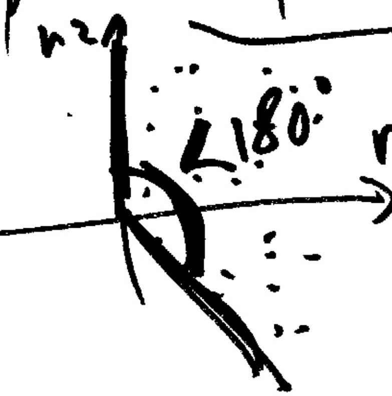
$h(n_1, n_2)$

$\to n=2$

$y(1) \longleftarrow \frac{1}{a} y(2) - \frac{b}{a} y(0) - \frac{1}{a} x(2)$



non recursive



recursively computable

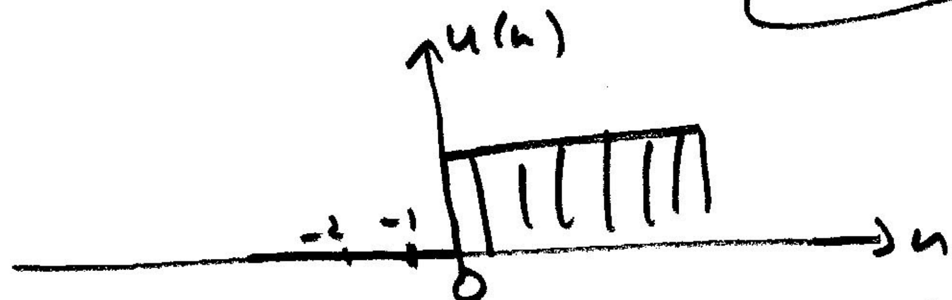
How to solve LCCDE with IRC or FRC using Z.T.

D.E : corresponds to a causal system.

Ex $y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = \underbrace{\left(\frac{1}{4}\right)^n u(n)}_{x(n) = \text{input.}}$

$$y(n) \leftarrow \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + \underline{\left(\frac{1}{4}\right)^n u(n)}$$

- $y(-1) = 0$
- $y(-2) = 0$



$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$Y(z) = \frac{6}{1 - \frac{1}{2} z^{-1}} + \frac{-8}{1 - \frac{1}{3} z^{-1}} + \frac{3}{1 - \frac{1}{4} z^{-1}}$$

ROC is chosen To compute I.Z.T. outside of some circle.

$$y(n) = 6\left(\frac{1}{2}\right)^n u(n) - 8\, u(n)\left(\frac{1}{3}\right)^n + 3\, u(n)\left(\frac{1}{4}\right)^n$$

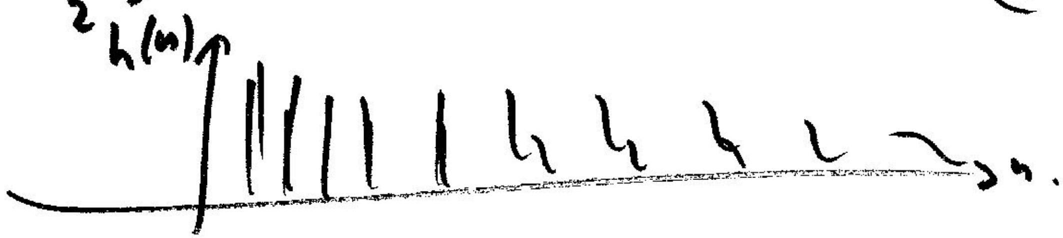
- How about Transfer fn?

$$Y(z) - \frac{5}{6} z^{-1} Y(z) + \frac{1}{6} z^{-2} Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}} = H(z)$$

ROC outside some circle of radius $\frac{1}{2}$

$$h(n) = 6\left(\frac{1}{2}\right)^n u(n) - 8\, u(n)\left(\frac{1}{3}\right)^n$$



Realizations of IIR filters with Rational Transfer fn :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

Assume causality:

$$y(n) \leftarrow \sum_{k=1}^{p} a_k \, y(n-k) + \sum_{k=0}^{q} b_k \, x(n-k)$$

Assume $h(n)$ is real, a_k, b_k are real.

Introduce This notation flow graphs:

① $x(n) \xrightarrow{\ c\ } c\,x(n)$ Scaling. or mult by c.

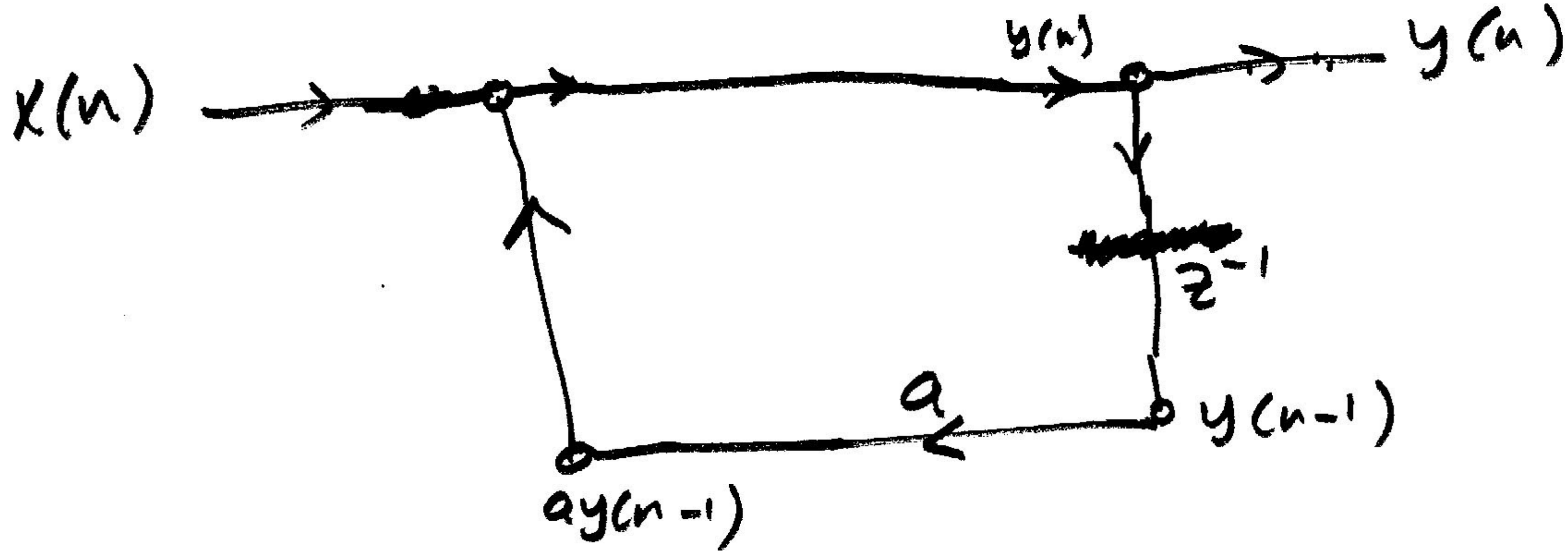
② Add 2 signals.

$$\begin{array}{l} x(n) \searrow \\ \qquad\quad \longrightarrow x(n) + y(n) \\ y(n) \nearrow \end{array}$$

③ Delay $x(n) \xrightarrow{\ z^{-1}\ } x(n-1)$

Q what is The flow graph for This causal system

$$y(n) \longleftarrow a\,y(n-1) + x(n)$$



Designing IIR:

1. specs.
2. Design $h(n)$
   Rational Transfer a_k, b_k
3. Realization
4. Implementation → ASIC, General purpose SP., PC, FPGA

4 methods for realization:

1. Direct form 1
2. Direct form 2
3. Cascade
4. Parallel.

Direct Form 1

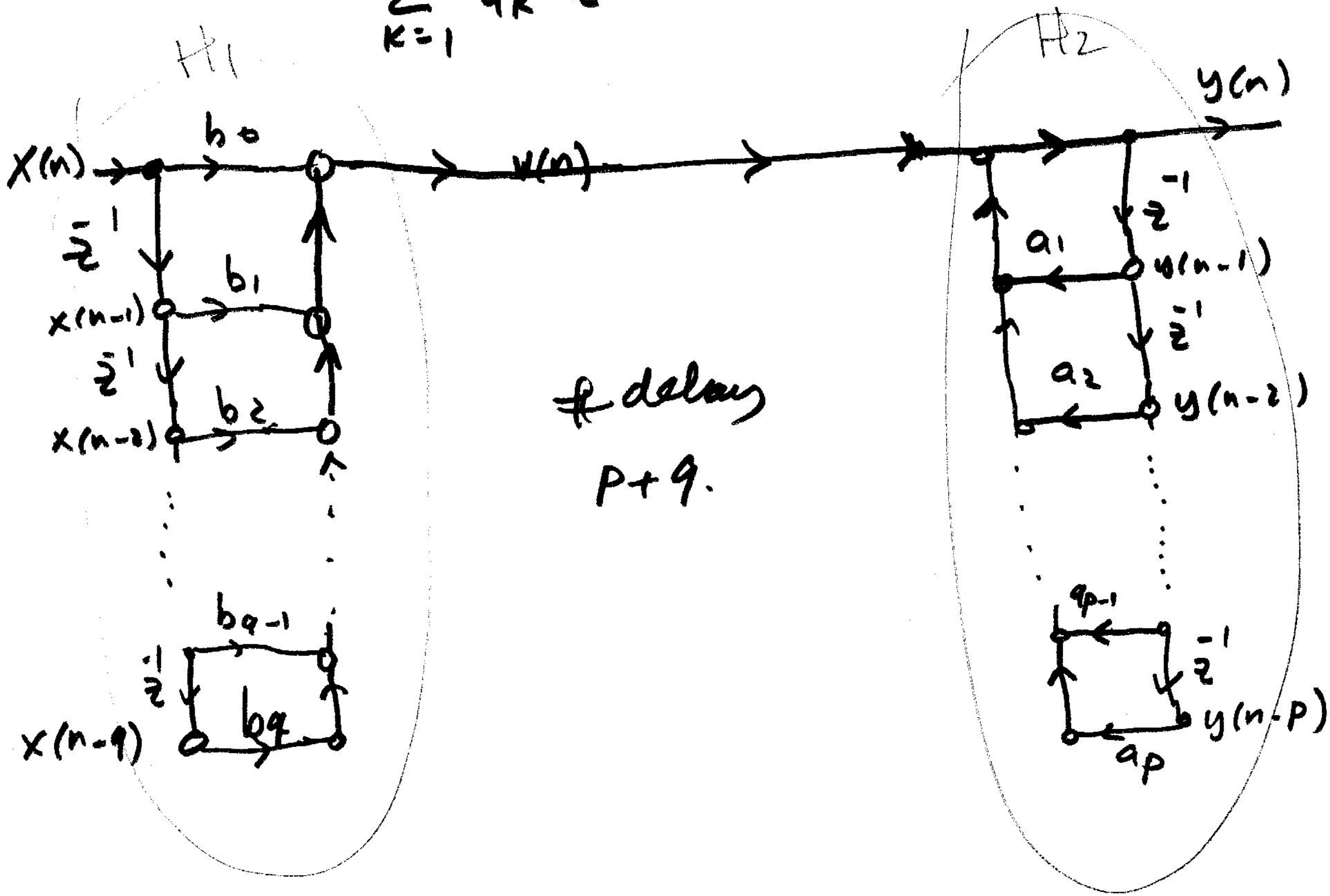
$$y(n) \leftarrow \sum_{k=1}^{p} a_k\, y(n-k) + \underbrace{\sum_{k=0}^{q} b_k\, x(n-k)}_{v(n)}$$

$b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_q x(n-q)$

$$y(n \leftarrow \sum_{k=1}^{p} a_k\, y(n-k) + v(n)$$

$$y(n) \leftarrow a_1 y(n-1) + a_2 y(n-2) + \cdots + a_p y(n-p) + v(n)$$

$$Y(z) = \sum_{k=1}^{p} a_k z^{-k} Y(z) + V(z)$$



\# delay

p+q.

$$x(n) \longrightarrow \boxed{H_1} \xrightarrow{v(n)} \boxed{H_2} \longrightarrow y(n)$$

$$H_1(z) = \frac{V(z)}{X(z)} = \sum_{k=0}^{q} b_k z^{-k}$$

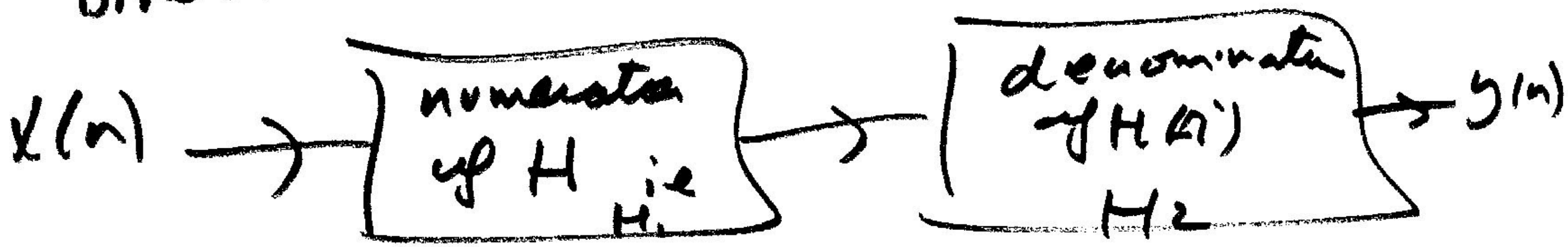
$$v(n) = \sum_{k=0}^{q} b_k \, x(n-k) \Longrightarrow \frac{V(z)}{X(z)} = \sum_{k=0}^{q} b_k z^{-k}$$

H_2: $$H_2(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

$$x(n) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

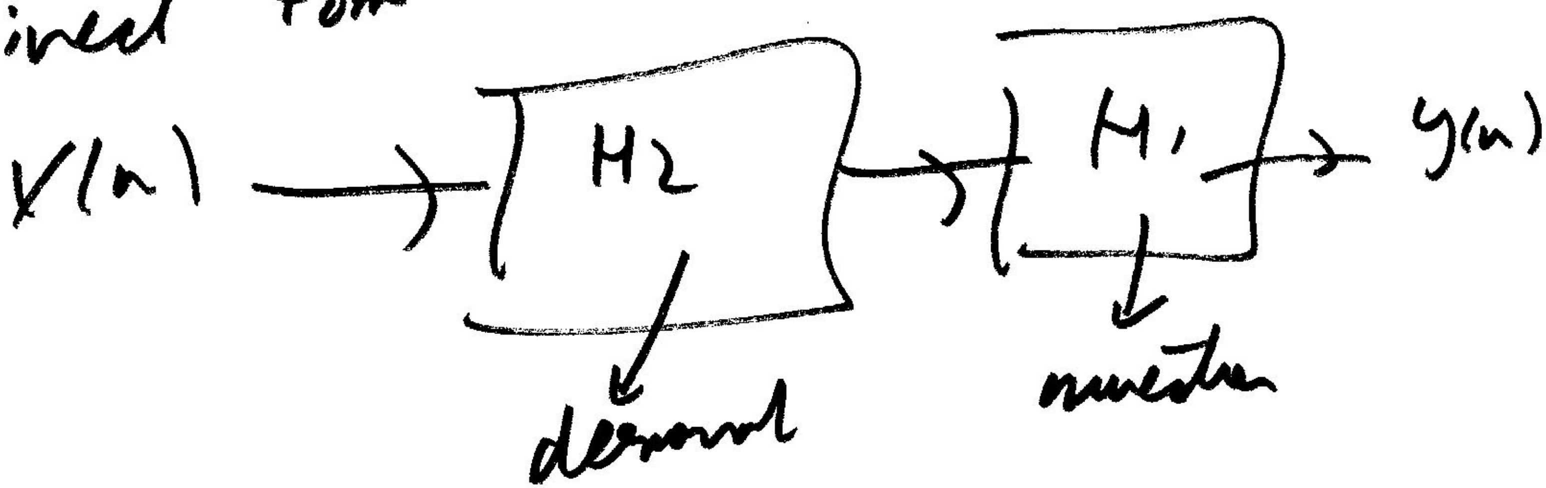
$$H(z) = H_1(z) H_2(z) = \frac{\overbrace{\sum_{k=0}^{q} b_k z^{-k}}^{H_1}}{\underbrace{1 - \sum_{k=1}^{p} a_k z^{-k}}_{H_2}}$$

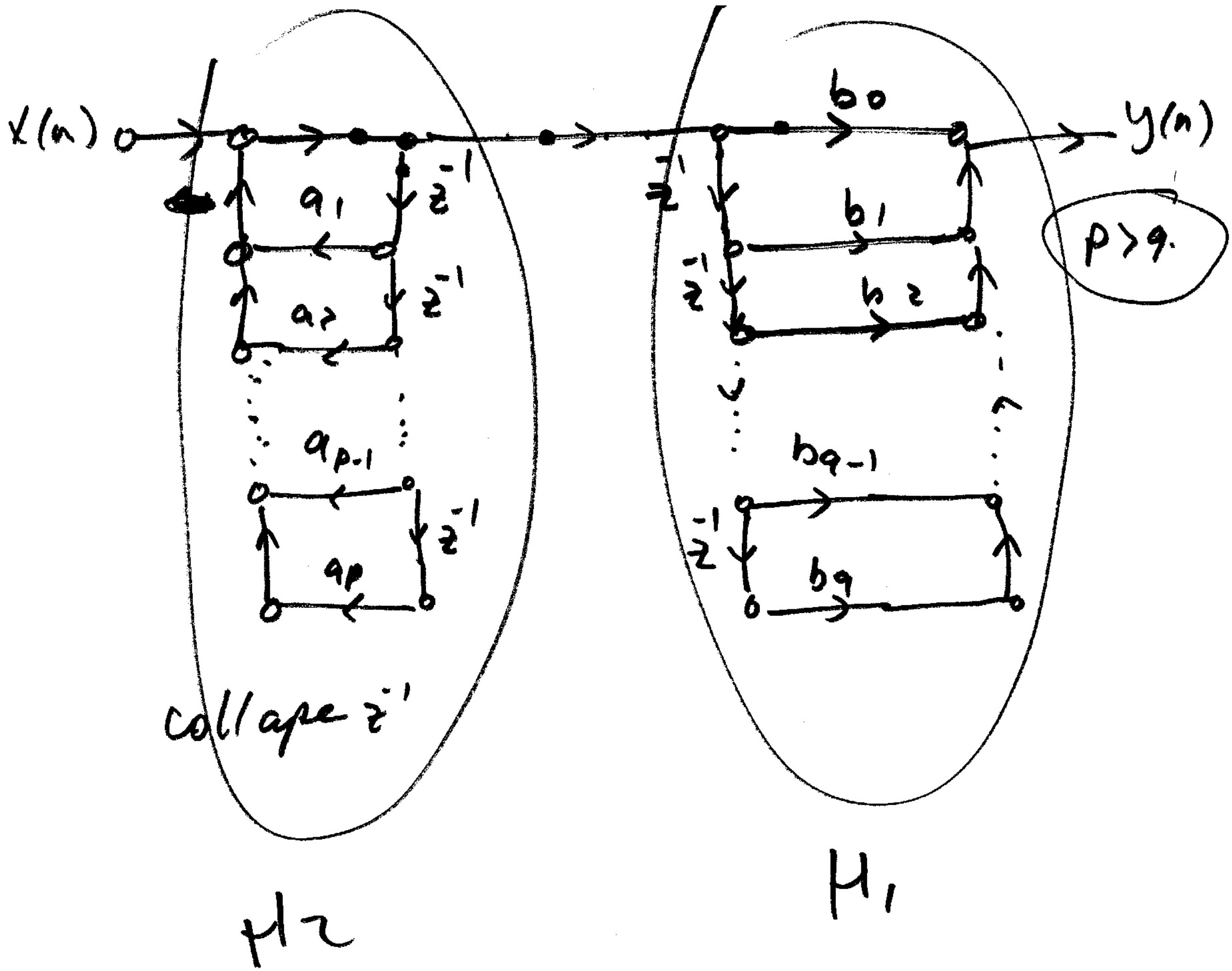
Direct Form 1.



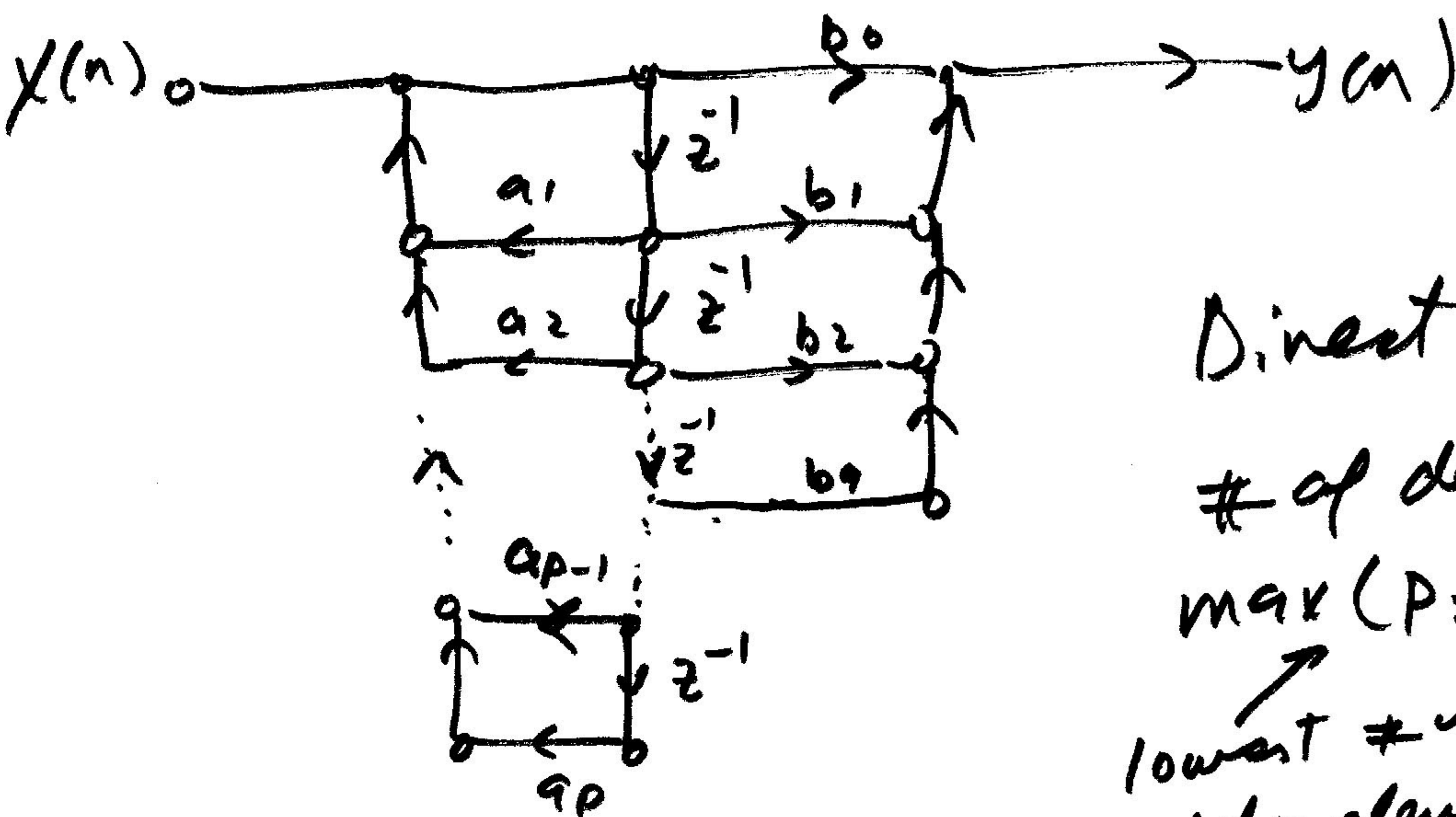
Preview

Direct Form 2





x(n)
y(n)
a_1
a_2
a_{p-1}
a_p
z^{-1}
b_0
b_1
b_2
b_{q-1}
b_q
$p>q$
collape z^{-1}
H2
H1



Direct form 2.

# of delay

$\max(p, q)$.

lowest # of possible delay elements to get away with.

canonical representation

Realizations of IIR filters with Rational Transfer function

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

1. Direct Form I
2. " " 2
3. Cascade structure
4. Parallel structure

Fundamental Thm of Algebra

Any polynomial in <u>one variable</u> can be factored into simple (1st order) Terms.

Polynomial of degree n, has n real or complex roots. Can be factored into n Terms.

<u>Ex</u> 3rd order (degree) polynomial.

$$P(z) = \alpha z^3 + \beta z^2 + \gamma z + \delta$$

$$= K \underbrace{(z - z_0)}_{\text{root 1}} \underbrace{(z - z_1)}_{\text{root 2}} \underbrace{(z - z_2)}_{\text{root \#3}}$$

ILL Conditioned → Problem
→ Algorithm.

<u>Problem</u> :

input data → $\boxed{\text{Problem}}$ → Answer
soln.

<u>Ex</u> : linear syst of Eqns. 2 equations in 2 unknowns

$$\left\{\begin{aligned} 5x + 3y &= 2 \\ 9x + 20y &= 15 \end{aligned}\right\} \quad x, y$$

$$A \quad x = b$$

<u>ALG</u> : Many diff. <u>algorithms</u> can be used to solve the <u>same</u> problem

1. Cramer's rule.
2. Gaussian elimination — good with pivoting; BAD without
3. Invert matrix. — ALG BAD
4. QR Decomposition — excellent
5. Gauss Seidel — good 3

Problem is ill conditioned: if small perturbation in the input results in large change in the output.

well conditioned prob ← $\begin{bmatrix} 5 & 3 \\ 9 & 20 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 15 \end{bmatrix}$

→ only dependent on the data in the problem itself not on the Alg used to solve it

det = 0 ill cond.! many soln ← Terrible ← $\begin{bmatrix} 5 & 3 \\ 10 & 6 \end{bmatrix}$ $\begin{bmatrix} 5 & 3 \\ 10 & 5.9999 \end{bmatrix}$ a little ill cond.

Condition # of a problem: Ratio between Relative change in output over relative change in input.

Condition # of problem is large ⟶ ill conditioned

" " " " " small ⟹ well conditioned
close to 1

- Problem itself can be inherently either
ill conditioned or well conditioned

~~If problem ill conditioned → you are stuck.~~

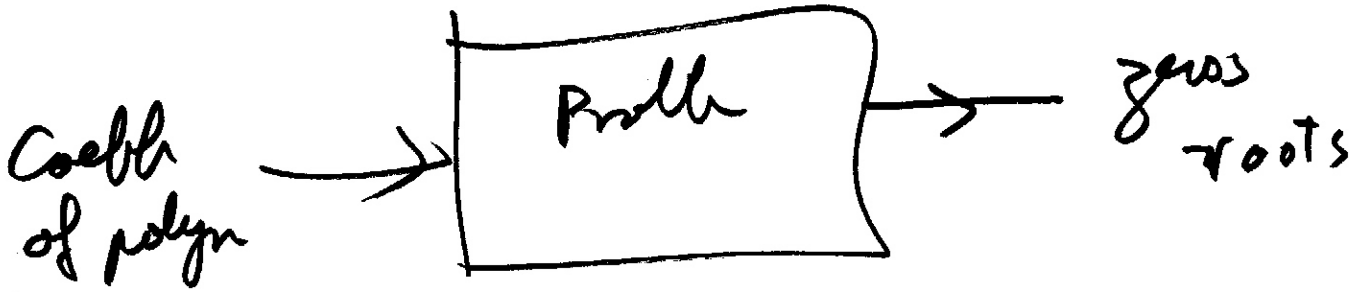
Condition # of The Alg

relative change in output over relative
change in the input using That alg
To compute input/output

If problem is wellconditioned, an ill conditioned
alg could give Bad results
But a well Conditioned Alg give good results

~~If~~

<u>Statement</u>: Problem of finding the roots of a polynomial is ill conditioned.



$$\alpha z^3 + \beta z^2 + \gamma z + \delta$$

- small perturbation in $\alpha, \beta, \gamma, \delta$ change the roots a lot.

- Problem is worse for higher deg. polynomials.

$$H(z) = \frac{P(z)}{Q(z)}$$

$\Rightarrow$ Direct Form 1 & 2 since they use coeffs of poly in their implementation.
The zeros and poles of the system are very sensitive to round off error.

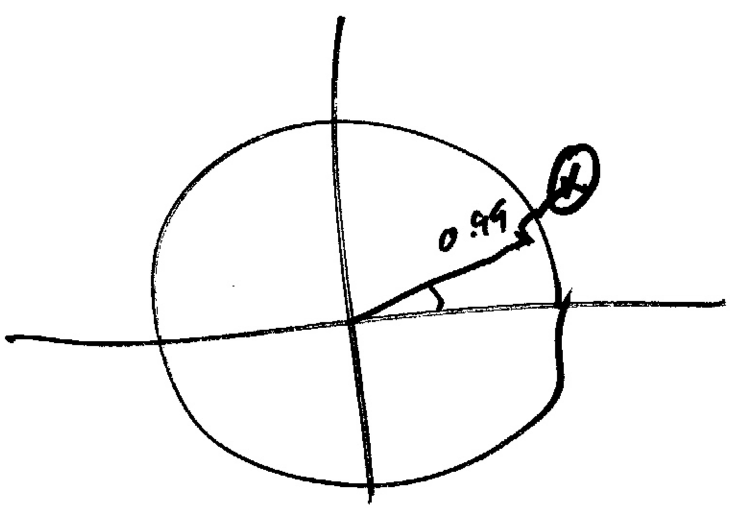
→ Finite precision in coeff representation

→ Finite precision in calculation.

↓

motivation

Cascade + Parallel.



Fundamental Thm of Algebra <u>does not</u> hold in 2 or 3 or higher # of variables...

$$P(z_1, z_2) = \alpha z_1^2 + \beta z_2^2 + \gamma z_1 z_2 + \delta + \epsilon z_2 + \eta z_2$$

<u>Thm</u>: set of factorable polynomials in 2 or more variable is of measure zero in the set of polynomials.

$\Rightarrow$ <u>Implication</u>: (1) cannot check stability of 2D. IIR filters easily.

(2) Cannot come up with cascade structure for 2D IIR.