Sampling Signals of Finite Rate of Innovation*

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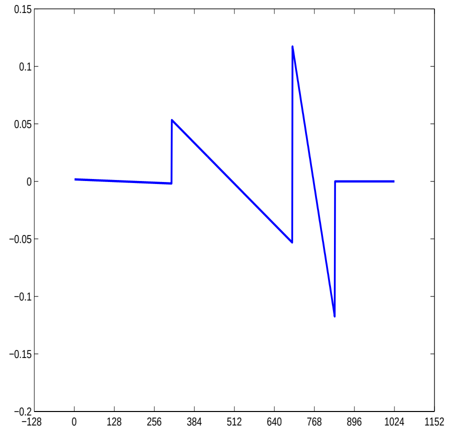

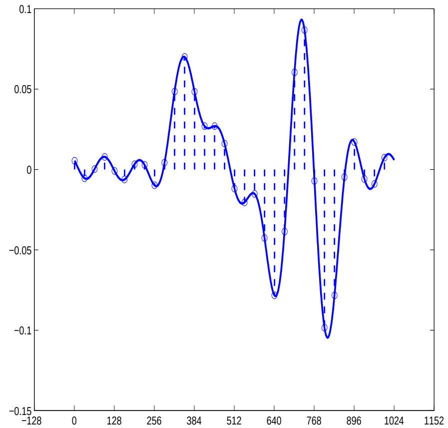

***Joint work with T. Blu, I. Maravic (EPFL), and P. Marziliano,(NTU)**

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3. **The Periodic Case**
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4. **Finite Length Signal Case**
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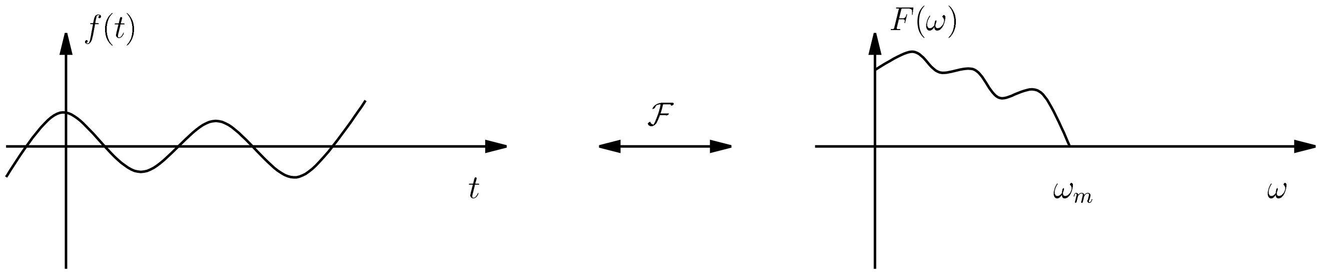
   - Radon transform
7. **Wideband communications**
   - CDMA and UWB
   - channel estimation
8. **Conclusions**

1 Motivation

Signal Processors love bandlimited Signals...



Then :

$$\{f(nT)\}, n \in Z, T = \pi/\omega_m$$

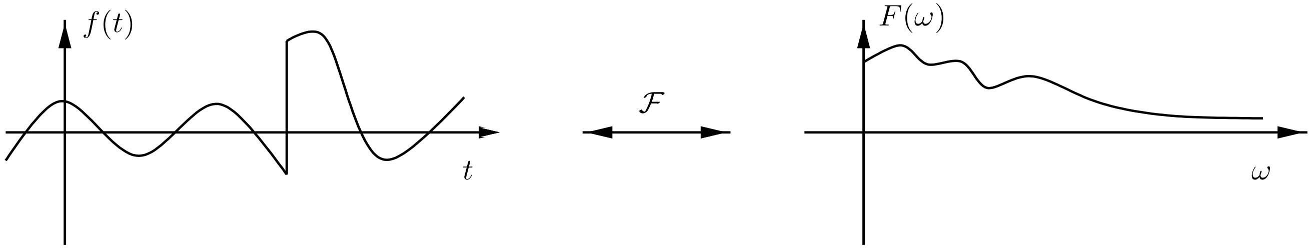
is a sufficient representation, since

$$f(t) = \sum_{n \in Z} f(nT)\,\mathrm{sinc}(t/T - n) \quad (1)$$

where

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} I[-\pi, \pi]$$

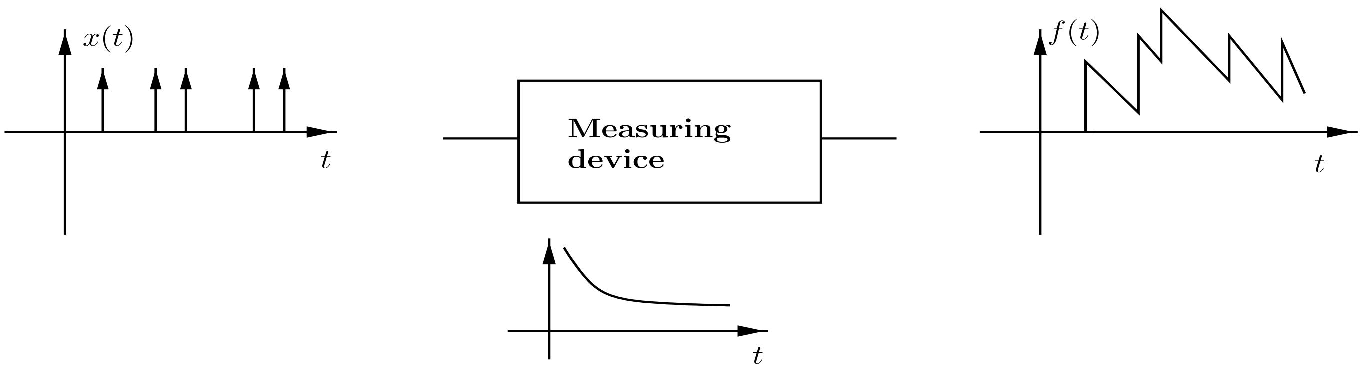
But what if



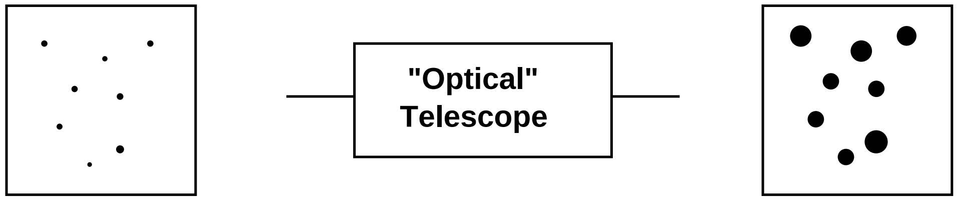
just one discontinuity and no more sampling theorem...

Often, one does not have access to the signal itself, but to a measurement

Example: neural spikes measured in non invasive manner ;)



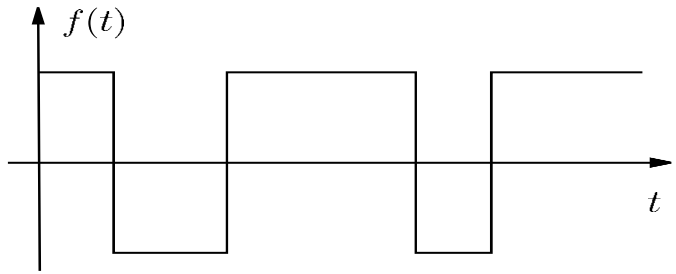
Example: photographing stars



Can we sample such signals that we see through an imperfect measuring device?

There are many parametric signals which are far from bandlimited

Example: CDMA



Note: rate of transition is finite, given by the chip rate symbol rate much slower

Example: Woodcut pictures

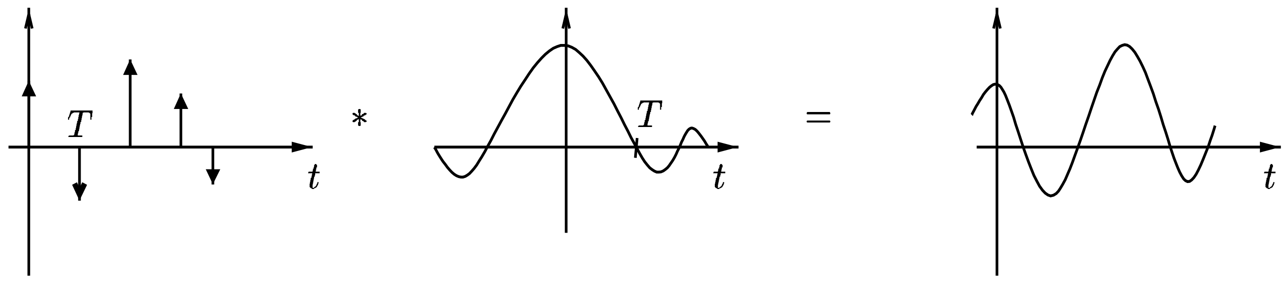
2 Signals of Finite Rate of Innovation

What is so special about a signal f(t) bandlimited to $[-\omega_m, \omega_m]$ **?**

With a sampling interval of $T = \pi / \omega_m$ **the signal f(t) is specified by**

$$\rho = 1/T = \omega_m/\pi$$

degrees of freedom per unit of time. By the interpolation formula (1), any bandlimited signal can be generated as



$\sum_{n\in\mathbb{Z}} f(nT)\delta(t-nT)$ $\quad$ $sinc(t/T)$ $\quad$ $f(t) = (1)$

Definition: The number of degrees of freedom per unit of time is called the rate of innovation ρ.

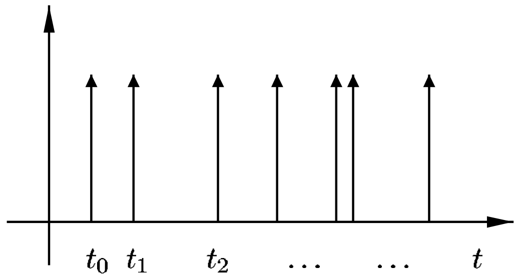
Rate of innovation

- Assume a class of signals having a parametric representation
- Consider one signal x from the class
- Call $C_x(t_0, t_1)$ the number of degrees of freedom in
- Then

$$\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_x\left(\frac{-\tau}{2}, \frac{\tau}{2}\right)$$

- If $\rho < \infty$, we call x a signal of **finite rate of innovation**

Example: Poisson process



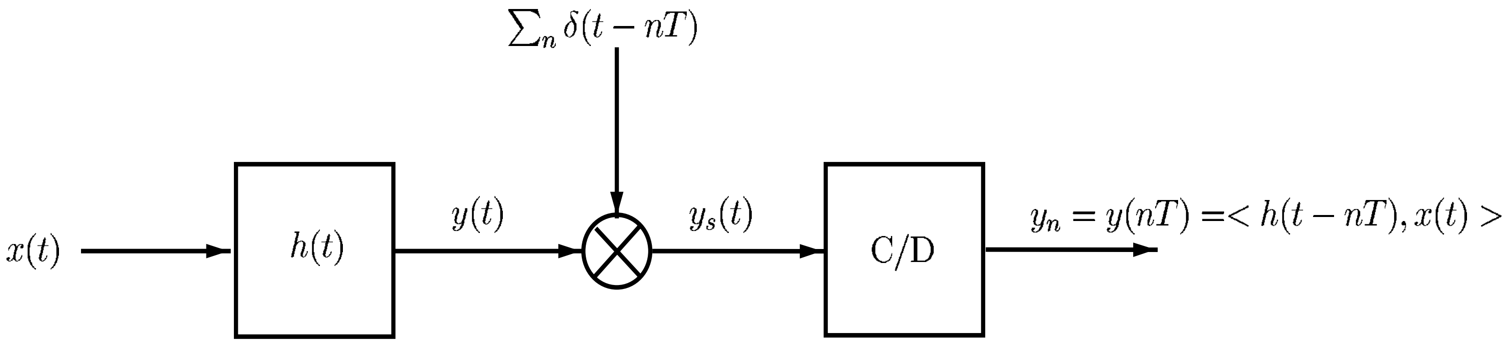
Interarrival times: i.i.d. , pdf $\mu e^{-\mu t}$

Expected interarrival time: $1/\mu$

$\{t_i\}$ **is a sufficient description of a realization**

$$\rho = \frac{1}{E(\text{int. time})} = \mu$$

Aquisition Model, Notation

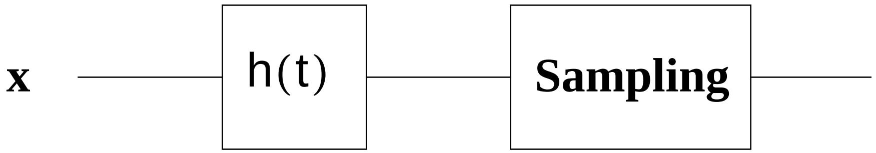


where x(t)**: signal**
h(t)**: sampling kernel**
y(t)**: filtered version of** x(t)
y_n**: samples**

Natural questions

1. What are interesting classes of signals with finite ρ

2. For which of these classes can we find unique representations through sampling (in particular uniformly) that is:



$$y_n = \langle h(t - nT), x(t) \rangle$$

such that $x \Leftrightarrow y_n$

just like in the bandlimited case

3. What are good kernels $h(t)$ **?**

4. What are the algorithms to find $x(t)$ **from** y_n **?**

1. “Classic”, subspace case. Given known fct $\varphi(t)$:

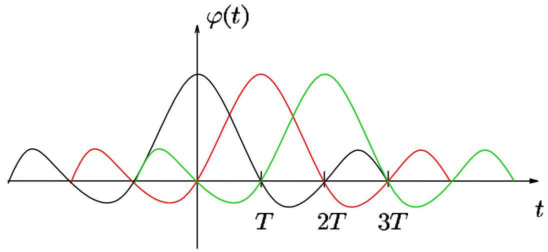
$$x(t) = \sum_{n \in Z} c_n \varphi\left(\frac{t}{T} - n\right)$$

Space: Span $\left\{\varphi\left(\frac{t}{T} - n\right)\right\}$

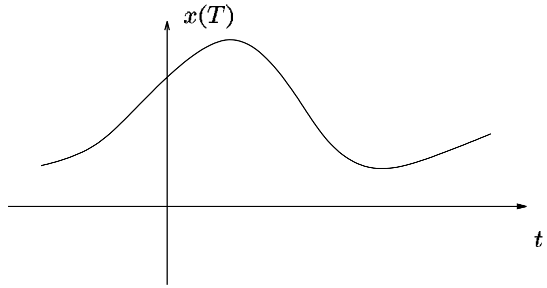
This is a well studied case (sampling, non-uniform sampling, reconstruction). It is a linear problem.

Example: Bandlimited signals $[-w_m, w_m]$**,** $\varphi(t) = \mathrm{sinc}(t)$

Basis:

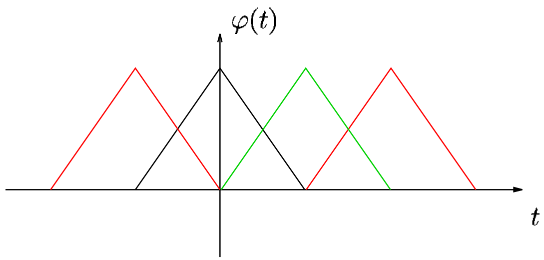


Ex:

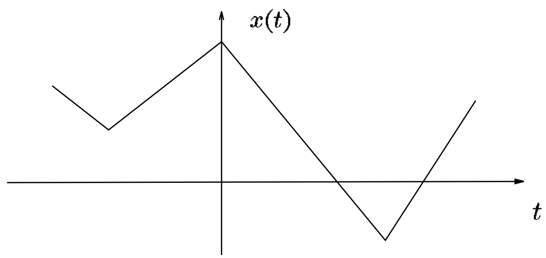


Example: Uniform, B-splines,

Basis:



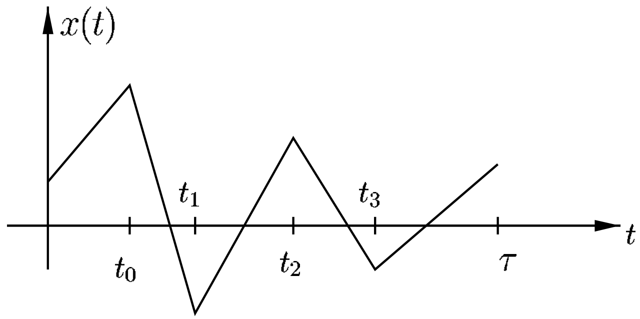
Ex:



2. Arbitrary shifts, known $\varphi(t)$ **:** $x(t) = \sum_{n \in Z} c_n \varphi\left(\frac{t}{T} - \tau_n\right)$

This is not a subspace!

Example: Non-uniform splines

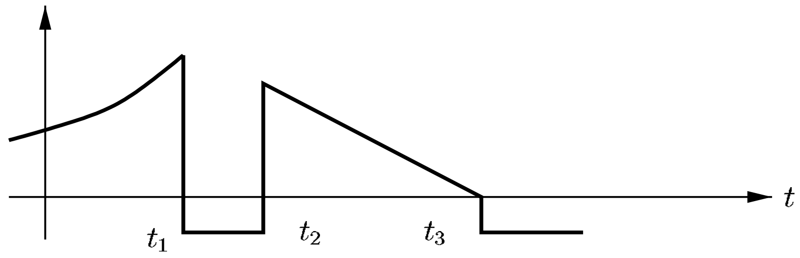


periodic non-uniform spline (deg. 1)

3. Arbitrary shifts, set of known fcts $\varphi_r(t)$:

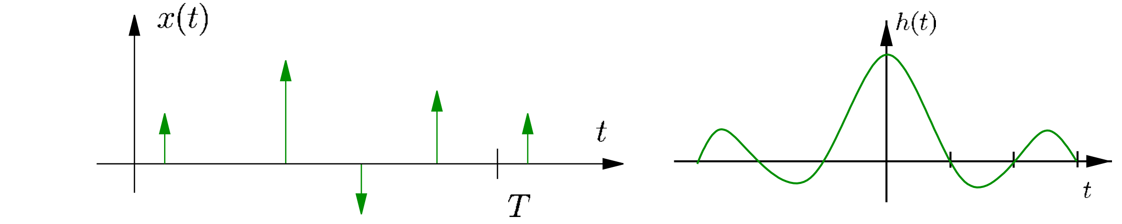
$$x(t) = \sum_{n \in Z} \sum_{r=0}^{R} c_{nr} \varphi\left(\frac{t - \tau_n}{T}\right)$$

Example: Non-uniform piecewise polynomials

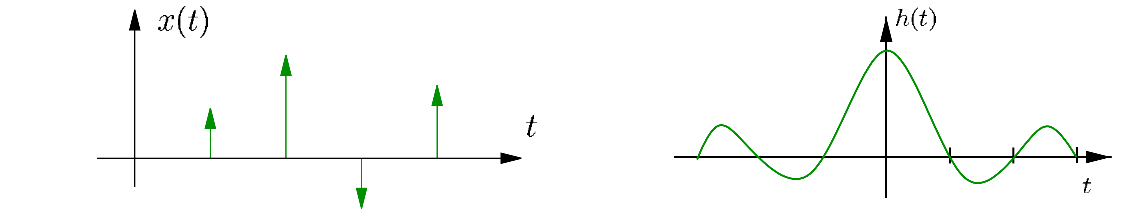


The 4 cases of interest:

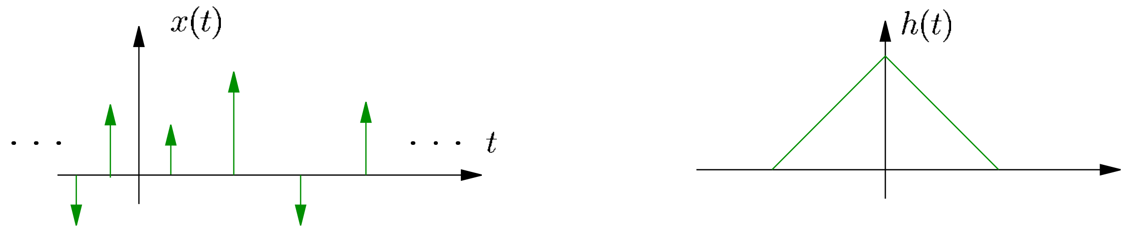
1 **Periodic signal infinite kernel**



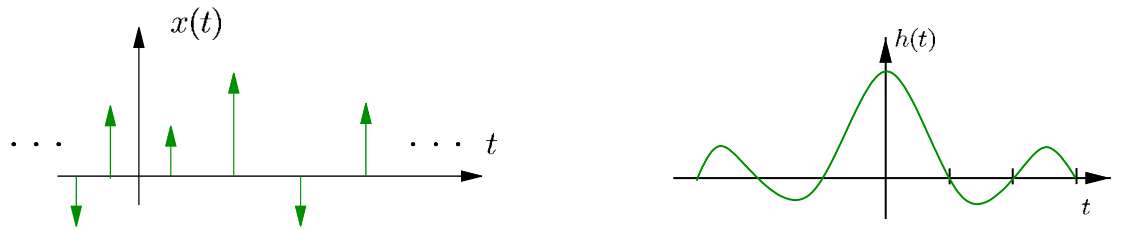
2 **Finite signal infinite kernel**



3 **Infinite signal finite kernel**

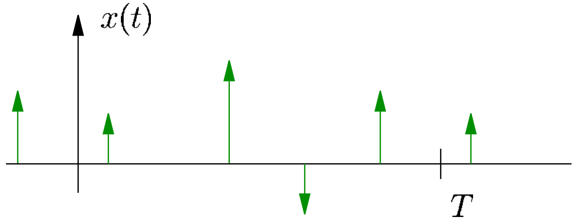


4 **Infinite signal infinite kernel**

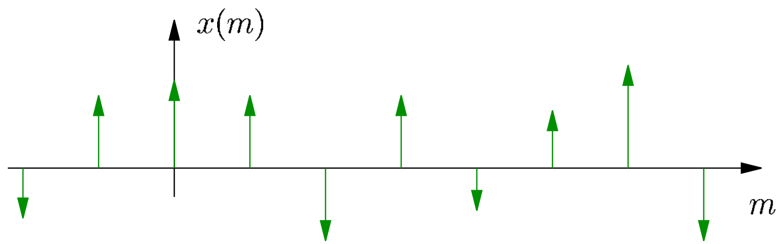


Note: 1, 2 and 3 lead to finite dimensional problems

3 The periodic case



Fourier series $$x(t) = \sum_{m \in Z} X[m] e^{\frac{j2\pi mt}{\tau}}$$



3.A Periodic “stream” of Diracs

K **Diracs per** τ **:**

$2K$ **degrees of freedom** $\rho = \frac{2K}{\tau}$

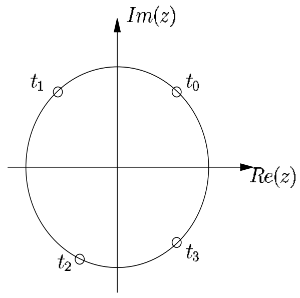
$$x(t) = \sum_{n \in Z} c_n \delta(t - t_n) = \sum_{n \in Z} \sum_{k=0}^{K-1} c_k \delta(t - t_k - n\tau) = \sum_{k=0}^{K-1} c_k \frac{1}{\tau} \sum_{m \in Z} e^{\frac{j2\pi m(t - t_k)}{\tau}}$$

or $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k e^{\frac{-j2\pi m t_k}{\tau}} \qquad m \in Z$

⚠ $X[m]$ **is a weighted sum of** K **exponentials** $\left(e^{\frac{-j2\pi t_k}{\tau}}\right)^m$

Consider:

$$A(z) = \sum_{m=0}^{K} A[m] z^{-m} = \prod_{k=0}^{K-1}\left(1 - e^{\frac{-j2\pi t_k}{\tau}} \cdot z^{-1}\right)$$



Now, note that

$$[1, -e^{\frac{-j2\pi t_k}{\tau}}] * [\ldots, e^{\frac{j2\pi t_k}{\tau}}, 1, -e^{\frac{-j2\pi t_k}{\tau}}, e^{\frac{-j4\pi t_k}{\tau}}, \ldots]$$

is zero, from which follows that $A[m] * X[m] = 0$

Equivalently, in time domain

$$a(t) = A(z)\Big|_{z = e^{\frac{-j2\pi t_k}{\tau}}} = \prod_{k=0}^{K-1}\left(1 - e^{\frac{-j2\pi(t_k - t)}{\tau}}\right)$$

has zeros at $t = t_k \quad k = 0, \ldots, K-1$**, thus** $a(t) \cdot x(t) = 0$

$A(z)$ **is called an annihilating filter, since it "kills"** $x(t)$

ECC: error locator polynomial

Theorem 1: Consider a periodic stream of K Diracs, of period τ, weights $\{c_k\}$ and locations $\{t_k\}$. Take a sampling kernel

$$h_\beta(t) = \beta \operatorname{sinc}(\beta t) \qquad \hat{\operatorname{sinc}} = I[-\pi, \pi] \qquad \text{where} \qquad \beta = \frac{2K + \frac{1}{2}}{\tau} > \rho$$

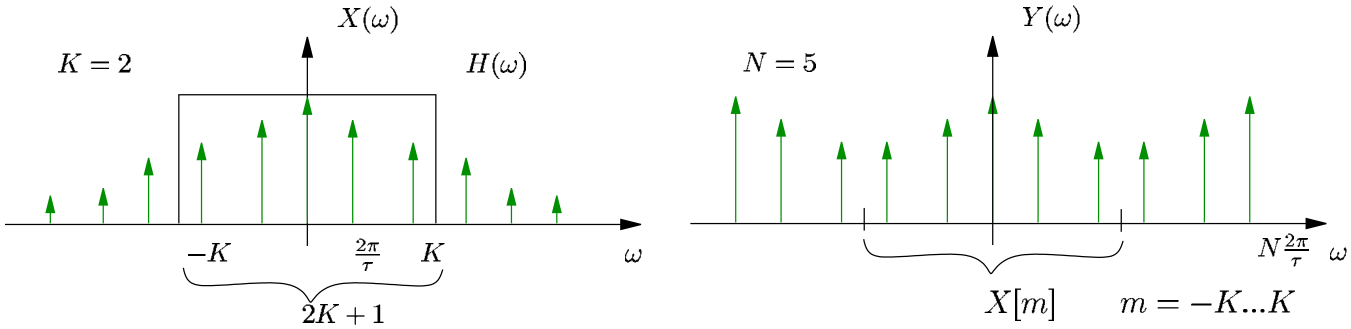
Pick $N = 2K + 1$ and $T = \tau / N$ Then

$$y_n = \langle h_\beta(t - nT), x(t) \rangle \text{ , } n = 0, \ldots, N-1$$

is a sufficient characterization of x(t).

Proof

1. y_n **is a sufficient characterization of** $X[m]$**,** $m = -K \ldots K$
Either use Poisson $\sum A[m] z^{-m}$**, or graphically:**



2. Finding $A[m]$ **s.t.** $A[m] * X[m] = 0$ $A[0] = 1$**, solve for** $m = 1 \ldots K$**. This leads to a Toeplitz system, e.g.** $K = 3$

$$\begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix} \begin{bmatrix} A[1] \\ A[2] \\ A[3] \end{bmatrix} = - \begin{bmatrix} X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

Classic Yule-Walker system
Unique solution for distinct Dirac locations

3. Factorisation of $A(z)$: $A(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1})$

where $u_k = e^{\frac{-j2\pi t_k}{\tau}}$, thus $\{t_k\}_{k=0}^{K-1}$ is found

4. Finding the weights c_k.

Given $\{t_k\}$, K values of $X[k]$ are given,

for ex. for $K = 3$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & 1 & 1 \\ u_0 & u_1 & u_2 \\ u_0^2 & u_1^2 & u_2^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

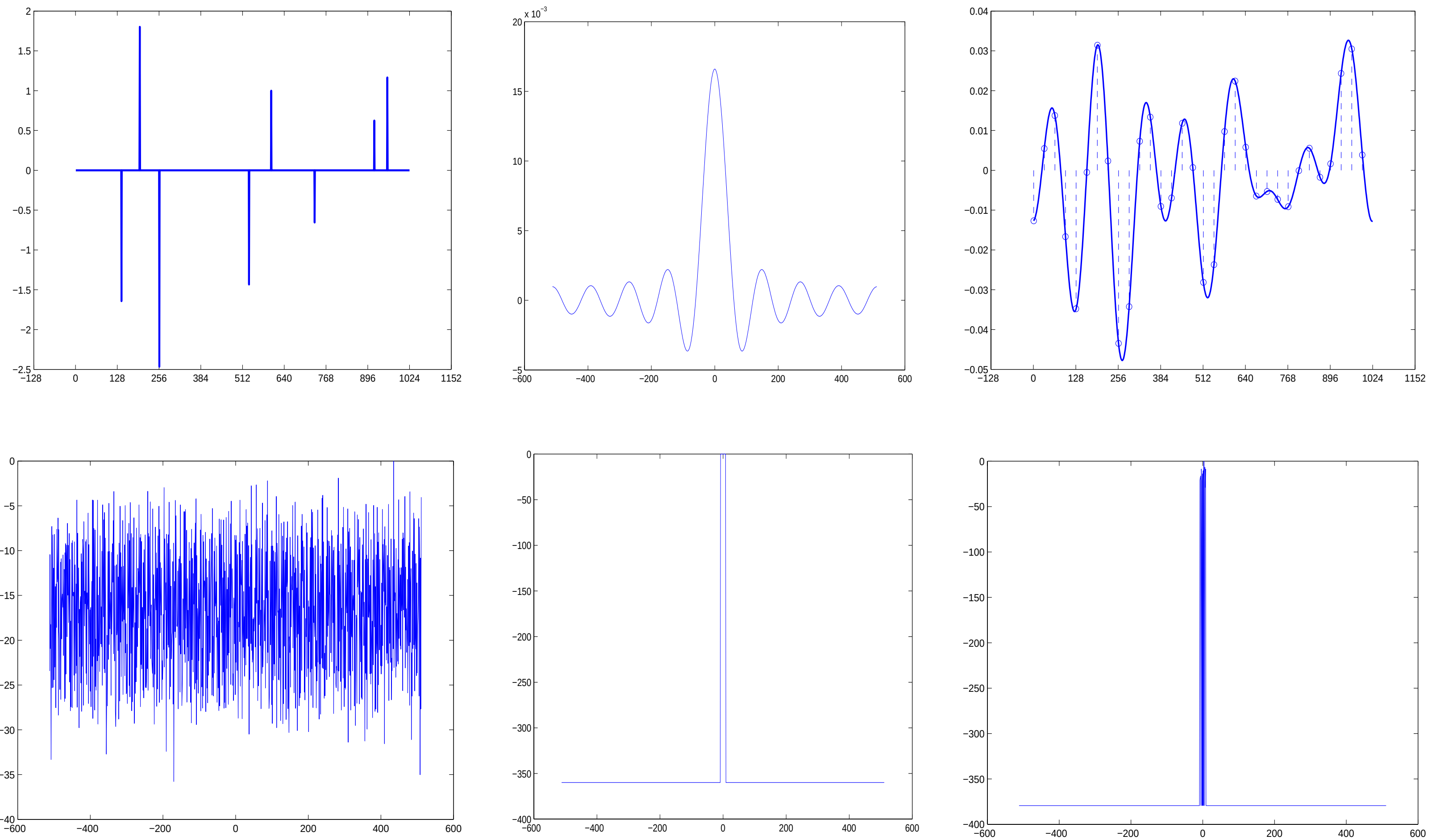
which is a Vandermonde system, having always a solution given distinct t_k's.

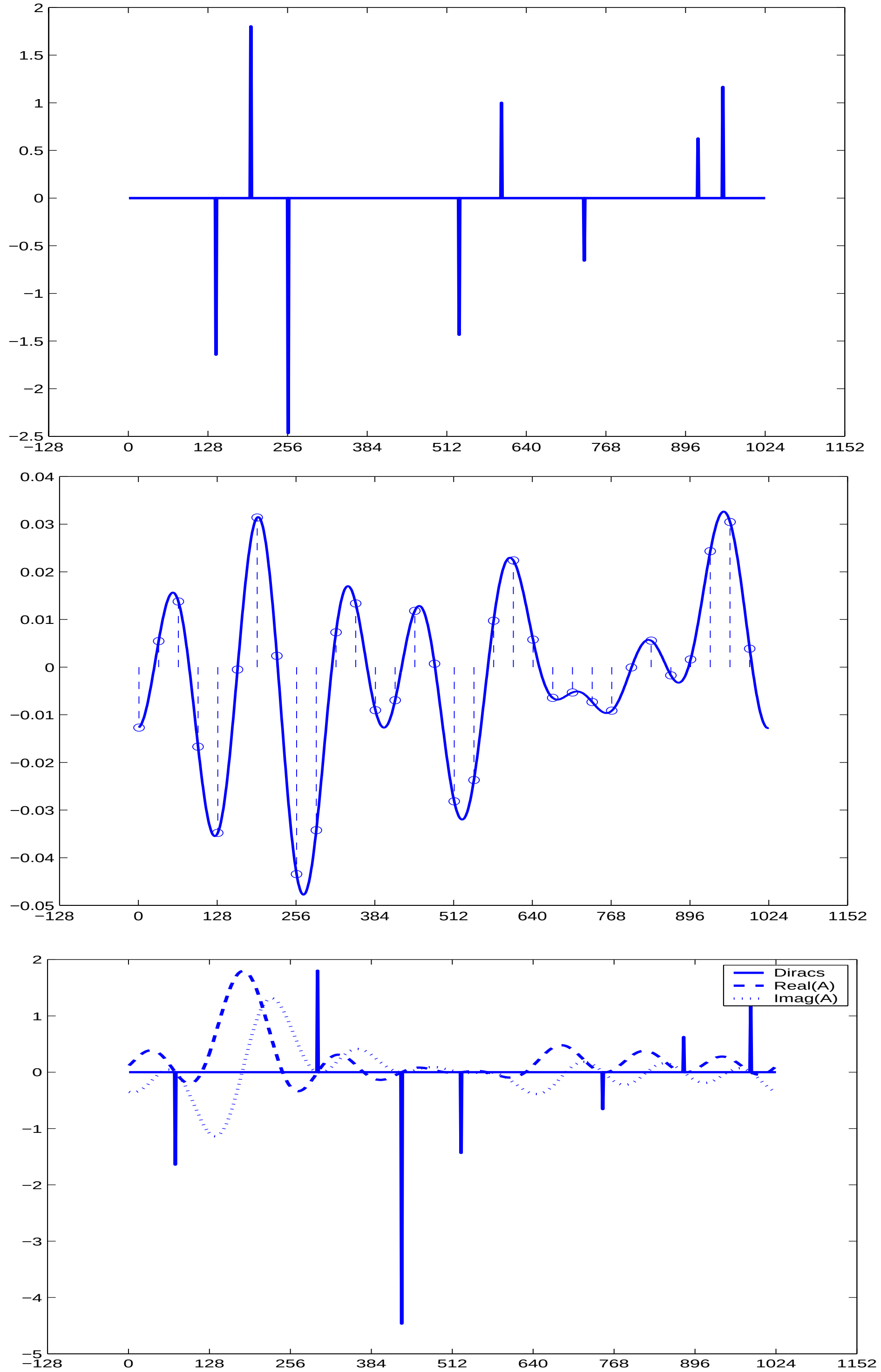
K=8

x(t) | h(t) | $y(t), y_n$

Time

Freq




Diracs
Real(A)
Imag(A)
−128 0 128 256 384 512 640 768 896 1024 1152

Interpretation

The projection of $x(t)$ **onto the lowpass space** $BL\left[\frac{-K2\pi}{\tau}, \frac{K2\pi}{\tau}\right]$ **is one-to-one for a periodic stream of K Diracs**

Corollary 1: Given $A[m],\ m = 0\ldots K$ and $X[m],\ m = -K\ldots K$ one can recover the entire spectrum as

$$X[m] = -\sum_{k=1}^{K} A[k]X[m-k] \quad , m = K+1\ldots$$

Proof: left to the reader

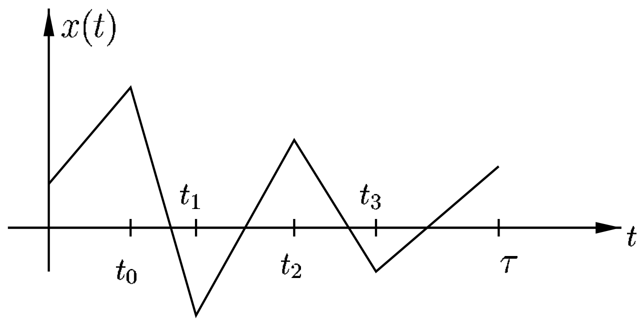
Notes: **1. annihilating filter known in sinusoidal retrieval from noise**

2. same filter used in error correction coding, and called error locator polyn

3. recursive spectrum extrapolation known as Berlekamp-Massey algo. in ECC

2,3 over finite fields...

3.B Non-uniform splines



periodic non-uniform spline (deg. 1)

A signal $x(t)$ **is a periodic non-uniform spline of degree** R **with** K **knots at** $\{t_k\}_{k=0}^{K-1}$ **iff its** $(R+1)^{th}$ **derivative is periodic of the form**

$$x^{(R+1)}(t) = \sum_{m \in Z} c_m \delta(t - t_m)$$

where $t_{m+k} = t_m + \tau$

Clearly, the Fourier series satisfy

$$X^{(R+1)}[m] = \left(\frac{j2\pi m}{\tau}\right)^{R+1} X[m] \qquad (*)$$

Thus

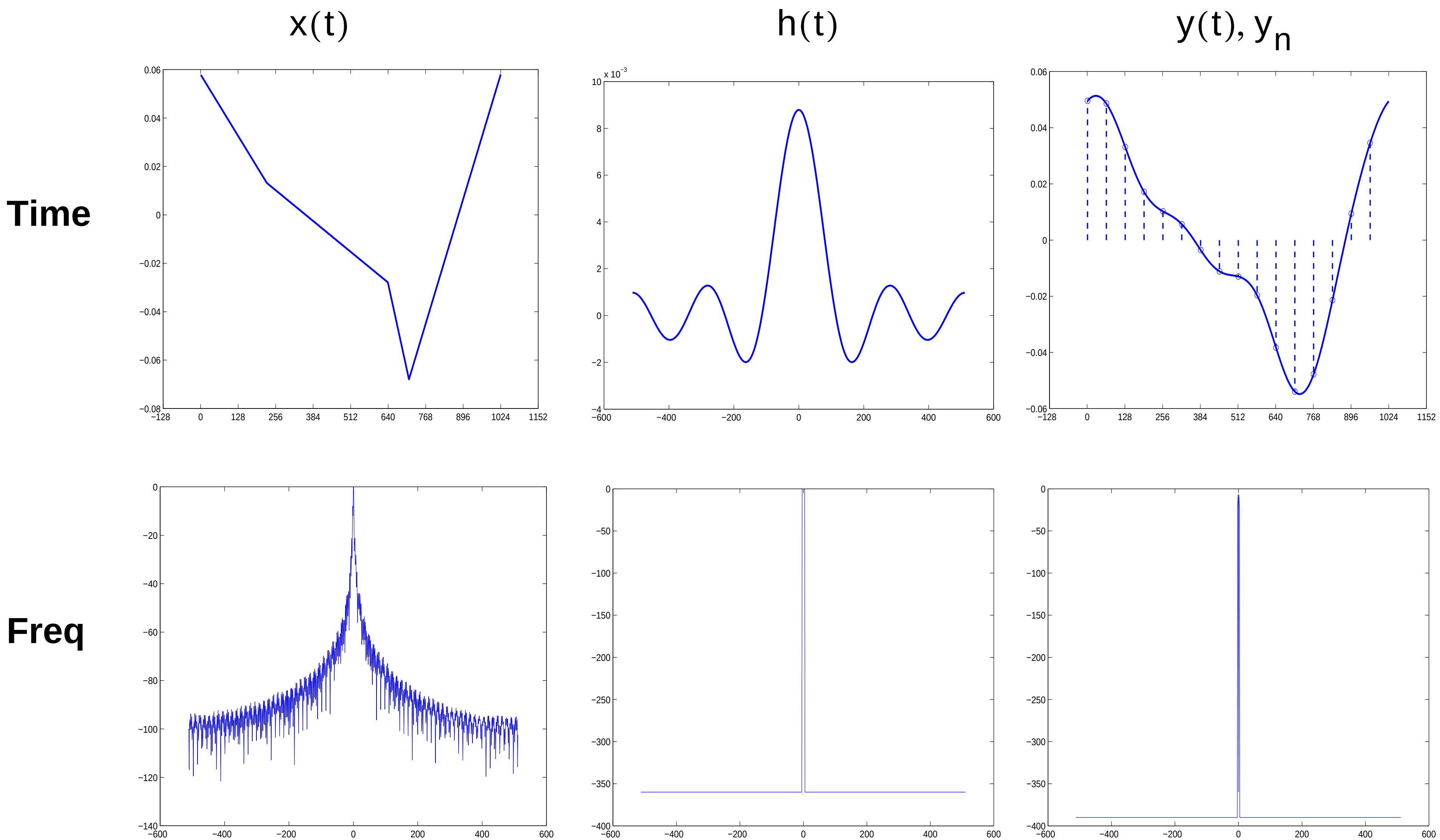
Theorem 2: Consider a periodic non-uniform spline of max degree R and period τ. Take $h_\beta(t)$ as sampling kernel, with

$$\beta = \frac{2K+1}{\tau} \quad \textbf{and} \quad T = \frac{\tau}{N} \qquad N = 2K+1$$

Then $\quad y_n = \langle h_\beta(t-nT), x(t)\rangle \qquad\qquad n = 0\ldots N-1$

uniquely defines $x(t)$.

Proof: similar to Thm 1 to get $X[m]$**. Then** $X^{(R+1)}_{[m]}$ **follows from (*), to which we apply Thm 1.** $X[0]$ **is added at the end**


x(t)
h(t)
y(t), y_n
Time
Freq

3.C Derivatives of Diracs

$$\delta^{(r)}(t) \ : \ \int f(t)\delta^{(r)}(t-t_0)dt = (-1)^r f^{(r)}(t_0)$$

where f is r**-times differentiable**

Then a periodic stream of differentiated Diracs is

$$x(t) = \sum_{m \in Z} \sum_{r=0}^{R_m - 1} c_{mr} \delta^{(r)}(t - t_m)$$

There are: K **locations,** $\tilde{K} = \sum_{k=0}^{K-1} R_k$ **weights.**

Thus: $\rho = \frac{K + \tilde{K}}{\tau}$

It can be verified that:

$$X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} \sum_{r=0}^{R_m - 1} c_{kr} \left(\frac{j2\pi m}{\tau}\right)^r e^{\frac{-j2\pi m t_k}{\tau}}$$

The annihilating filter now requires multiples zeros, since $(1-u_k z^{-1})^R$ **annihilates** $m^{R-1} u_m^k$ **. Thus** $A(z)$ **becomes**

$$A(z) = \prod_{k=0}^{K-1} (1-u_k z^{-1})^{R_k}$$

Then: $A[m]*X[m] = 0$**, therefore, one can show:**

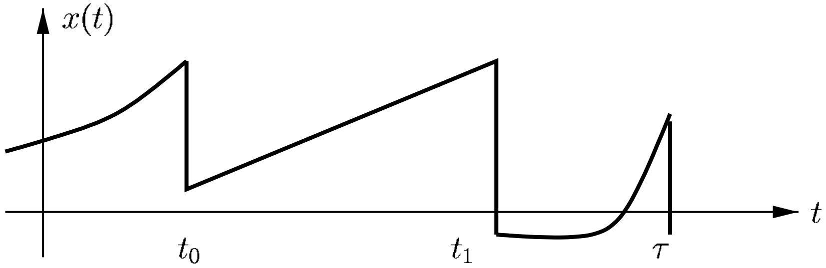
Theorem 3: Consider a periodic stream of differentiated Diracs as above. Take as sampling kernel $h_\beta(t) = \beta \operatorname{sinc}(\beta t)$ with $\beta = \rho + 1/\tau$ and sample $h_\beta \tilde{S} x$ at N points $t = n\tau/N$ where $n = 0 \ldots N-1$ and $N = K + \tilde{K} + 1$. Then

$$y_n = \langle h_\beta\left(t - n\frac{\tau}{N}\right), x(t)\rangle \qquad n = 0 \ldots N-1$$

is a sufficient characterization of $x(t)$.

Proof: Similarly to Thm 1, we first get $X[m]$ **from** y_n**.**
Then we solve for the location $\{t_k\}$ $A[m]*X[m] = 0$ **and finally for the coefficients** $\{c_{kr}\}$**.**
The latter calls for a generalized Vandermonde system which is non-singular for $t_i \neq t_j \quad i \neq j$.

3.D Piecewise Polynomials



A periodic piecewise polynomial $x(t)$ **with** K **pieces of degree max** R **has an** $(R+1)^{th}$ **derivative which is a stream of differentiated Diracs, or**

$$x(t) = \sum_{m \in Z} \sum_{r=0}^{R_m - 1} c_{mr} \delta^{(r)}(t - t_m)$$

There are: K **locations,** $\tilde{K} = (R+1)K$ **weights**

$$\rho = \frac{(R+2)K}{\tau}$$

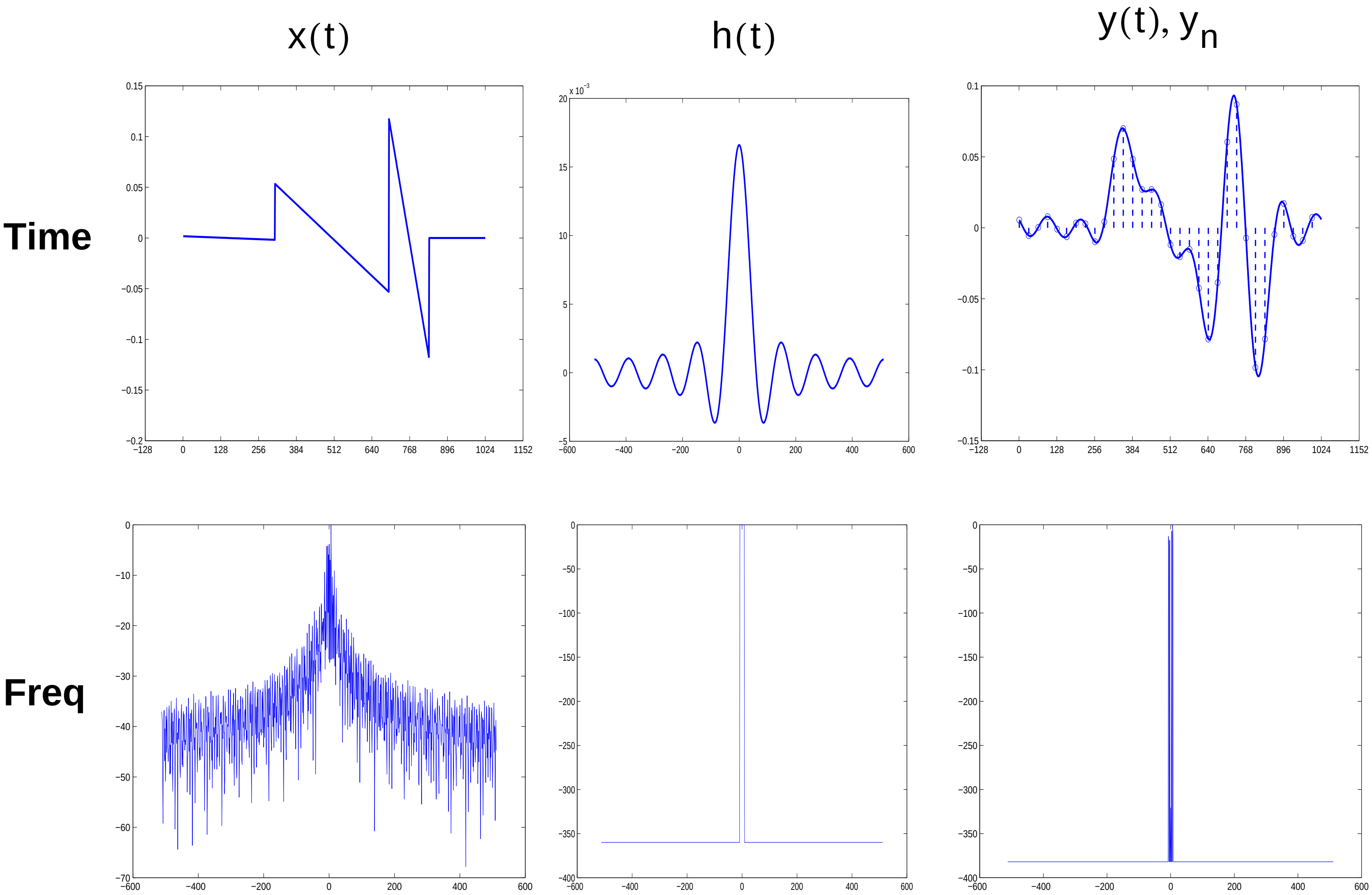
Then:

Theorem 4: A signal defined by its derivatives as in (**) can be recovered after convolution by $h_\beta(t)$, where $\beta = \rho + 1/\tau$ and sampling at $t = n\tau/N$ with $N = (R+2)K+1$, that is

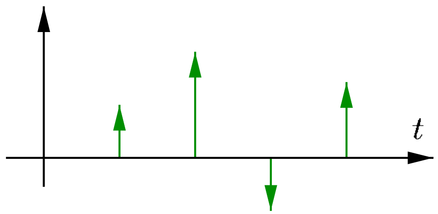
$$y_n = \left\langle h_\beta\left(t - n\frac{\tau}{N}\right), x(t)\right\rangle \qquad n = 0 \ldots N-1$$

uniquely specifies $x(t)$

Proof: left to the reader, along Theorem 1, 2 and 3.


x(t)
h(t)
y(t), y_n
Time
Freq
x 10^-3

4 Finite Length Signals



A finite length signal with finite ρ clearly has a finite # of degrees of freedom.

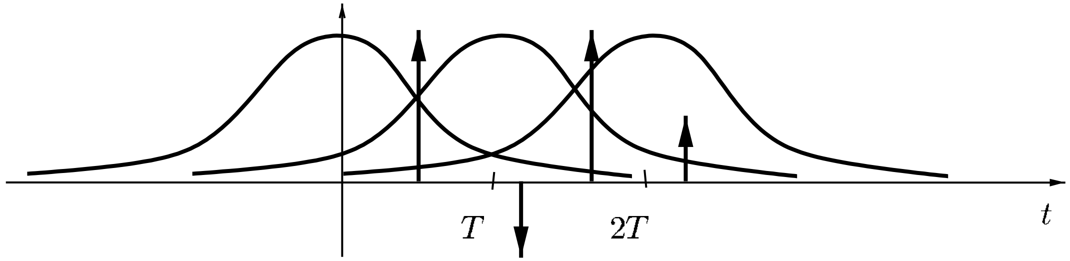
The question of interest is:

> given a sampling kernel with a **infinite support** (like the sinc or the gaussian), is there a **finite set of samples** that uniquely specifies the signal?

4.1 Gaussian Kernel

Consider the same signal as in (4.1), now using a gaussian kernel

$$h(t) = e^{-t^2/(2\sigma^2)}$$



Then, the sample values are

$$y_n = \left\langle x(t), e^{-\left(\frac{t}{T} - n\right)^2/(2\sigma^2)} \right\rangle$$

$$y_n = \sum_{k=0}^{K-1} c_k e^{-\left(\frac{t_k}{T} - n\right)^2/(2\sigma^2)}$$

Expanding (4.1)

$$y_n = \sum_k c_k e^{\frac{-t_k^2}{T^2 2\sigma^2}} \cdot e^{\frac{nt_k}{T 2\sigma^2}} \cdot e^{\frac{-n^2}{2\sigma^2}} \qquad (4.2)$$

Introduce

$$Y_n = e^{\frac{n^2}{T 2\sigma^2}} \cdot y_n$$

Thus

$$Y_n = \sum_k \underbrace{c_k e^{\frac{-t_k^2}{T^2 2\sigma^2}}}_{a_k} \cdot \underbrace{\left(e^{\frac{t_k}{T 2\sigma^2}} \right)^n}_{u_k^n}$$

$$Y_n = \sum_{k=0}^{K-1} a_k u_k^n \qquad (4.4)$$

that is ... a linear combination of exponentials!

Therefore, use the usual method of the good old annihilating filter

$$A * Y = 0$$

and factor it such as to find $\{u_k\}_{k = 0 \ldots K-1}$

From u_k**:**

$$t_k = 2\sigma^2 T \ln u_k$$

From u_k **and** t_k **and** K **values of** Y_n**, we can solve for** c_k **in (4.2). Thus**

Theorem 5: Given a finite stream of K Diracs and a gaussian kernel $h(t) = e^{-t^2/(2\sigma^2)}$, then N samples

$$y_n = \langle x(t), h\left(\frac{t}{T} - n\right)\rangle$$

where $N \geq 2K$, are sufficient to reconstruct the signal.

Note: Similar remarks as for Theorem 3...

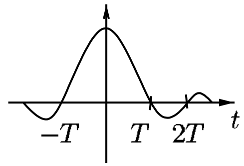
But: Here, unlike in the sinc case, we have an "almost local" reconstruction because of the exponential decay of $h(t)$ **!**

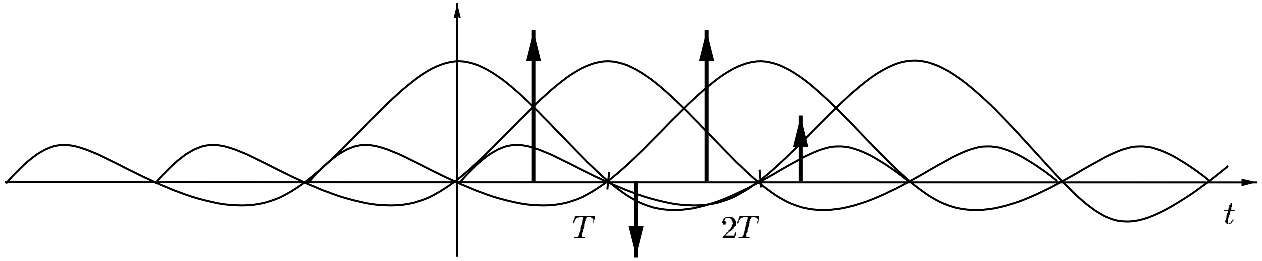
4.2 Sinc kernel (Thierry's tour de force)

Consider a finite sequence of spikes

$$x(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k) \qquad (4.5)$$

and a kernel $\mathrm{sinc}(t/T)$





The samples $y_n = \langle x(t), \mathrm{sinc}\left(\frac{t}{T} - n\right)\rangle$ **are**

$$y_n = \sum_{k=0}^{K-1} c_k \mathrm{sinc}\left(\frac{t_k}{T} - n\right) = (-1)^n \sum_{k=0}^{K-1} c_k \frac{\sin(\pi t_k / T)}{\pi\left(\frac{t_k}{T} - n\right)} \qquad (4.6)$$

Introduce the following interpolators:

$$P(u) = \prod_{k=0}^{K-1}\left(\frac{t_k}{T} - u\right) = \sum_{k=0}^{K} p_k u^k \text{ , \textbf{deg.} } K$$

$$P_l(u) = \prod_{k \neq l}\left(\frac{t_k}{T} - u\right) \text{ , \textbf{deg.} } K-1$$

Then, consider the following

$$Y_n = (-1)^n P(n) y(n) = \frac{1}{\pi}\sum_{k=0}^{K-1} C_k \sin((\pi t_k)/T) P_k(n) \quad (4.7)$$

$$Y = A \cdot C$$

Now (key insight!) Y_n **is of degree** $K-1$ **Thus**

$$\Delta^K Y_n = 0 \qquad n = K \ldots N-1 \qquad (4.8)$$

$$V \cdot p = 0 \qquad N - K \geq K$$

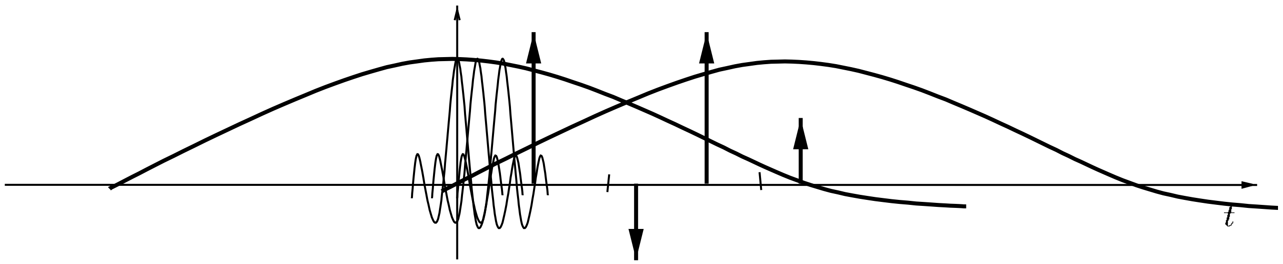
Note: ~Δ^k **similar to annihilating filter**

So, as long as $N - K \geq K$, one can use (4.4) to solve for P_k from y_n. This leads to $\{t_0, t_1, \ldots, t_{K-1}\}$.

Using this in (4.6) allows to solve for $\{c_i\}$. Thus:

Theorem 6: Given a finite stream of K Diracs and a $\mathrm{sinc}(t/T)$ kernel, N samples $y_n = \langle x(t), \mathrm{sinc}\left(\frac{t}{T} - n\right) \rangle_{n = 0 \ldots N-1}$ where $N \geq 2K$, are sufficient to reconstruct the signal.

Note: the result does not depend on T! of course, it shows up in the conditionning of linear system!!



The steps to reconstruct the signal are

1. **Solve a linear system** KxK

$$\{y_i\} \rightarrow \{p_i\}\ ,\ i = 0 \ldots K-1\ (p_K = 1)$$

2. **Factor**

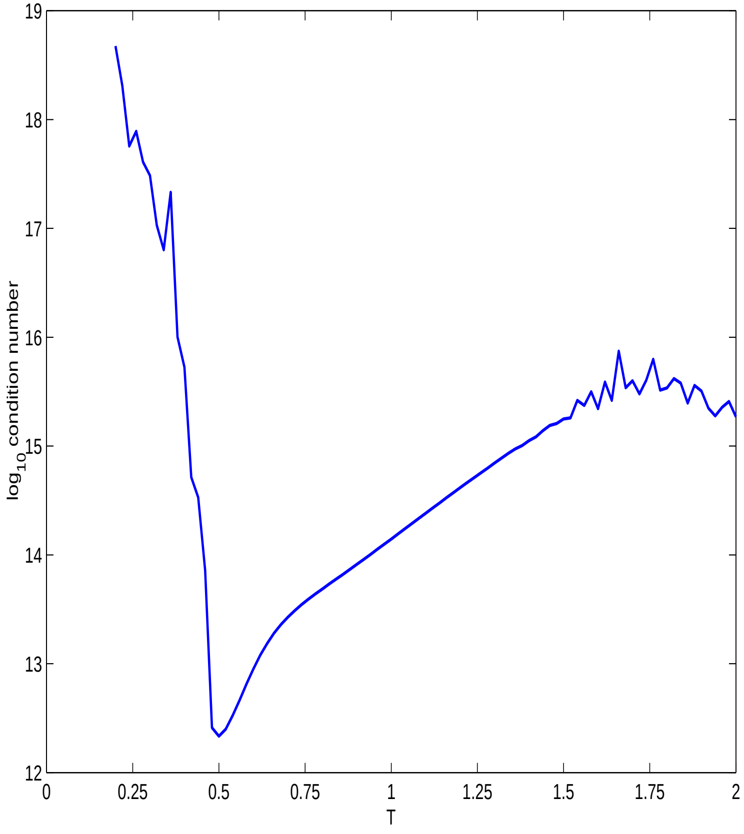
$$P(u) \rightarrow \{t_i\}\ ,\ i = 0 \ldots K-1$$

3. **Solve linear system** $\rightarrow \{c_i\}$

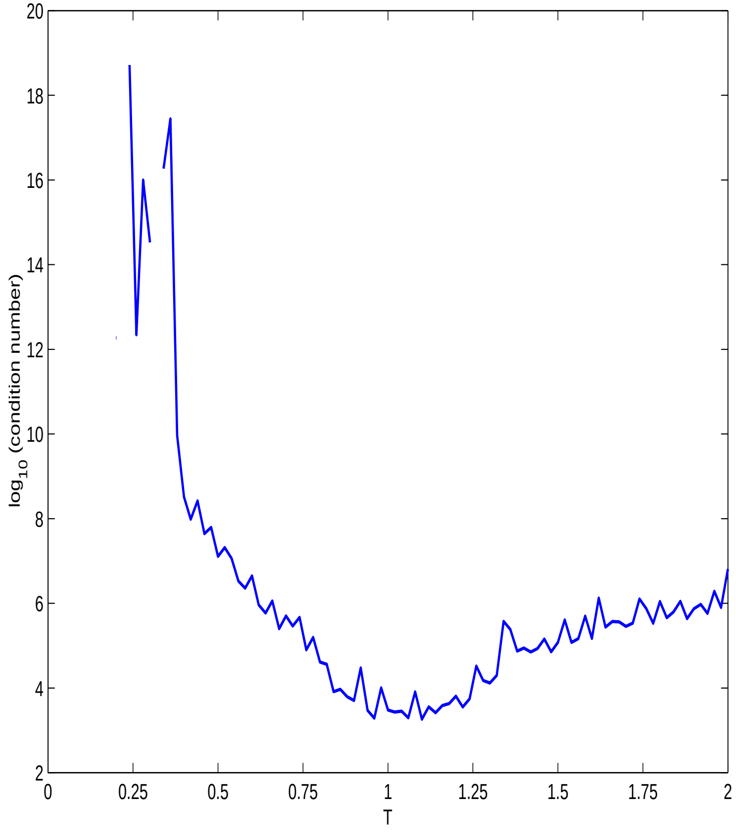
This method can be extended to piecewise polynomials, similarly to Theorem 4.

Also, there is an obvious equivalent for discrete-time signals from $l_2(Z)$ **and discrete-time sinc kernels.**

Sinc Kernel, finite length signals



Conditioning on location



Conditioning on weights

5. Applications

We show 2 direct applications of the results shown above.

5.1 Piecewise Bandlimited Signals

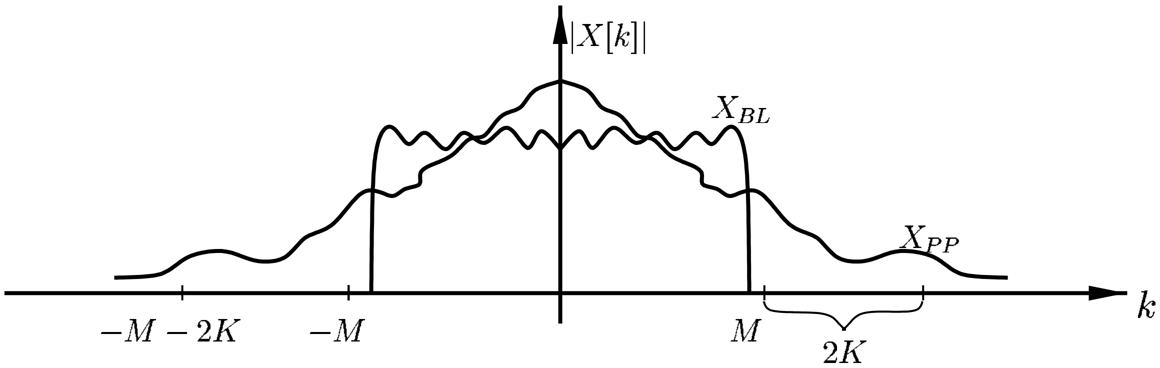
Consider a signal that is the sum

$$x = x_{BL} + x_{PP}$$

where x_{BL} **is bandlimited and** x_{PP} **is piecewise polynomial.**

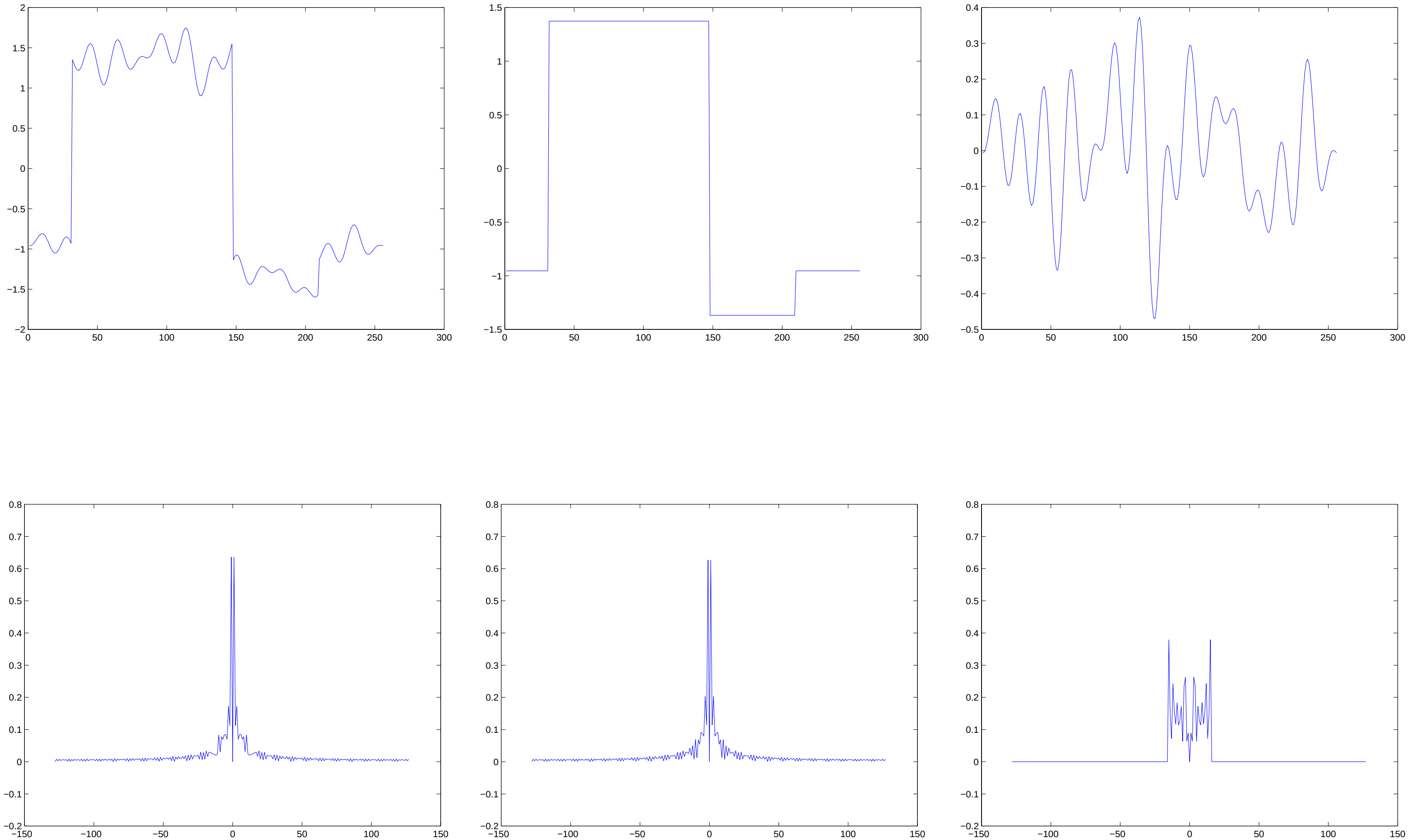
Assume x_{BL} **is specified by its frequency component** $X_{BL}[k]$ **,** $k \in [-M, M]$ **while** x_{PP} **has** $2K$ **degrees of freedom.**

Then, consider the spectrum of $X[k]$ **,** $k \in [-M-2K, M+2K]$**.**



First, using $X[k]$ **,** $k \in [M+1, M+2K]$ **and the technique of Proposition 1 or Theorem 1, we can recover** x_{PP}**. Substracting** X_{PP} **from** X**, we can then recover** X_{BL}**.**

Piecewise Bandlimited Signal



Thus:

Proposition 3: Given a piecewise BL signal of length N, with $2M+2K$ degrees of freedom. Pick Q a divisor of N and $\varphi[n] = \mathrm{IDTFS}(I[-2K-M, M+2K])$.

Then

$$y[l] = \langle x[n], \varphi[n-lQ] \rangle_{circ}$$

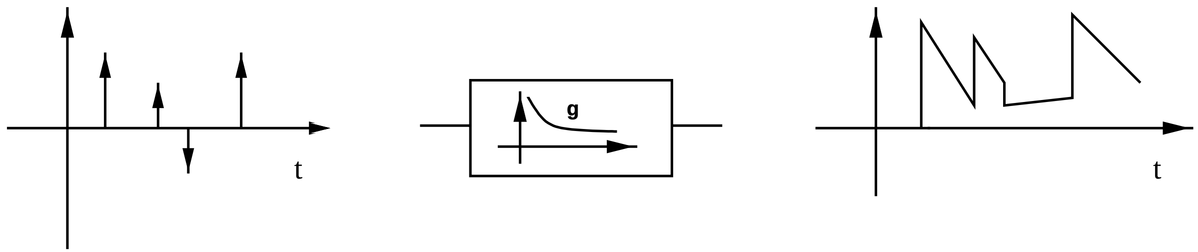
uniquely specify $x[n]$ if

$$\frac{N}{2Q} > M + 2K$$

The proof follows from earlier results with adjustements •

5.2 Filtered Piecewise Polynomials

Consider a stream of K **Diracs convolved with a known filter** $g(t)$

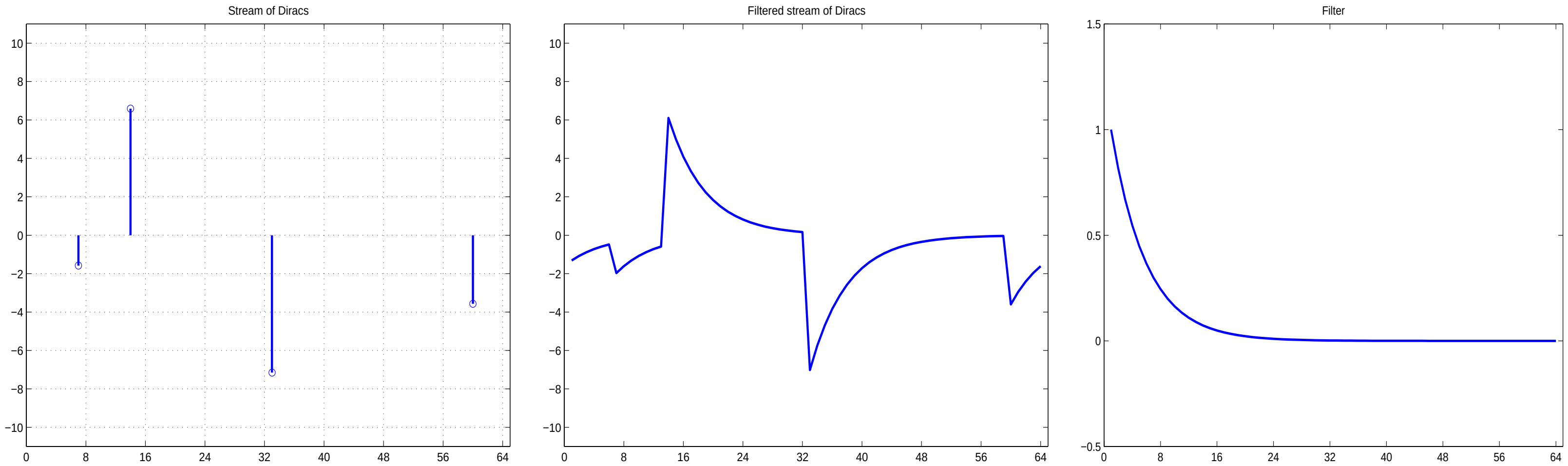


Thus: $x(t) = g(t) * d(t)$

where g **is known and** $d(t) = \sum_i \alpha_i \delta(t - t_i)$

Clearly, if $g[n] \leftrightarrow G[k]$ **is invertible over** $2K$ **frequency values, then we can use Proposition 1.**

Example:



Stream of Diracs
Filtered stream of Diracs
Filter
10
8
6
4
2
0
−2
−4
−6
−8
−10
1.5
1
0.5
0
−0.5
0
8
16
24
32
40
48
56
64

In particular:

Propositon 4: Assume $x[n]$ with K Diracs and a filter $G[k] \neq 0$, $k \in [-K, K]$. The signal we observe is $x[n] * g[n]$.

Using $\varphi[n] = IDTFS(I_{[-K, K]})$ and M such that $\frac{N}{2M} > K$, M a divisor of N

Then

$$y[l] = \langle x[n], \varphi[n - lM] \rangle$$

is a sufficient representation of $x[n]$.

A more difficult case appears when $g[n]$ is unknown but of finite ρ ...

6 Multidimensional Case

2D Poisson: K **Diracs on** R^2/T

Various approaches

- non separability is the key!
- $X[m_1, m_2]$, $|m_i| \leq K$ is sufficient $\Rightarrow O(K^2)$ samples
- $X[m_1, m_1]$, $|m_1| \leq K$ is sufficient $\Rightarrow O(K)$ samples

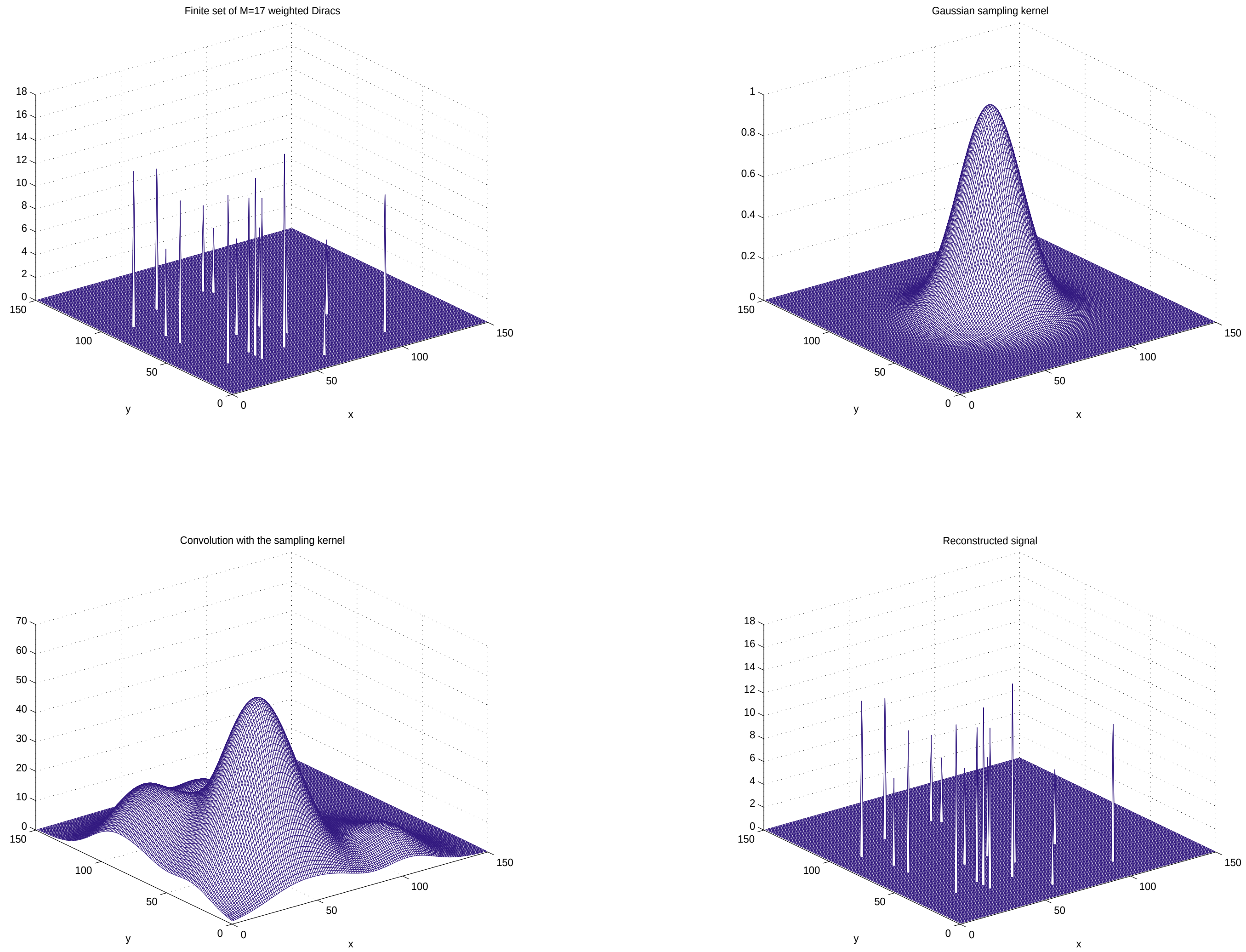
--> 2D root finding (...) or spectral extrapolation

Extension:

- lines
- simples objects

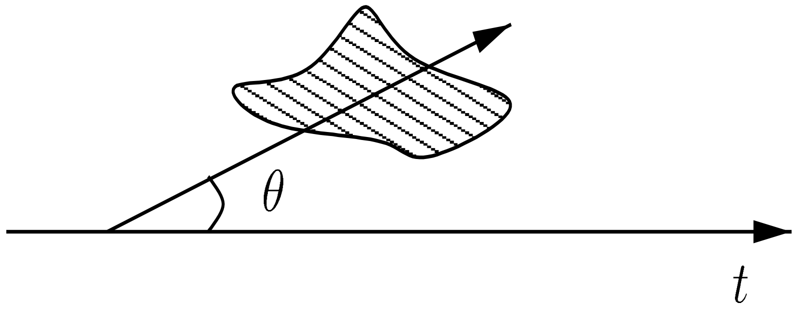
Goal: #samples ~ #deg. of freedom of object

Example of a 2D gaussian kernel:



2D methods based on projections

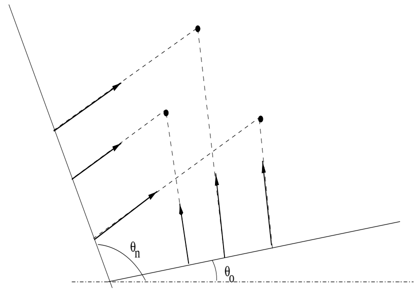
Radon Transform

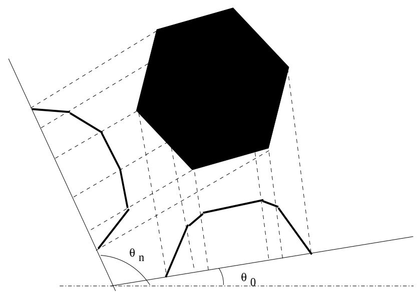


$$f(x, y) \Leftrightarrow F(\theta, t)$$

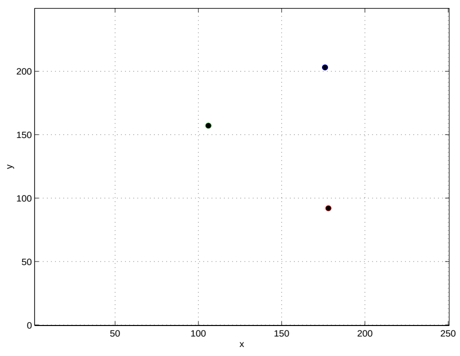
What about “finite complexity” objects?

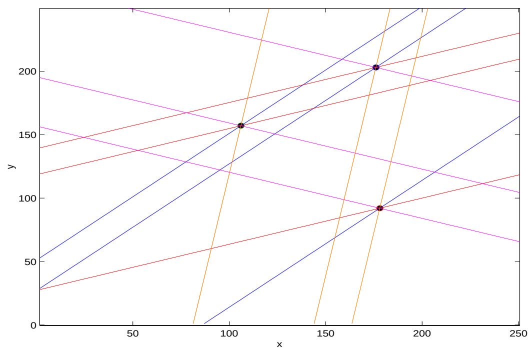
$\Rightarrow$ **Projections are finite rate of innovation!**





Result: Set of K Diracs can be perfectly reconstrucded from K+1 bandlimited projections with 2K samples



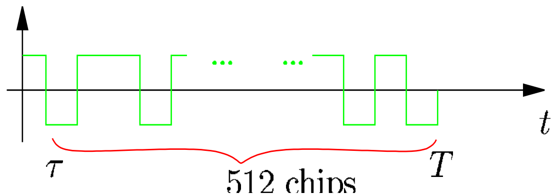


See [Maravic] ICASSP-2002

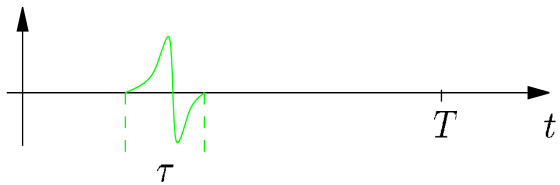
7 Communications Applications

Many communication systems use wideband signalling

CDMA: chip rate >> symbol rate



UWB: pulse position modulation



In both cases

rate of innovation << bandwidth

But: Noise !
Solution: oversample
subspace methods, SVD

7.1 Solving for sinusoids in noise

Idea: Solve for "longer" filter:

M+1xM+1

$$\begin{bmatrix} x(0) & x(-1) & \dots & x(-M) \\ x(1) & x(0) & & \\ x(2) & & \dots & \\ \dots & & & \\ x(M) & & \dots & x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ \\ a_{M-1} \end{bmatrix} = \begin{bmatrix} x(1) \\ \\ \\ \\ x(M+1) \end{bmatrix}$$

using 2M+1 samples > 2K **oversample**

Now: The noiseless Toeplitz matrix has rank K (# of sinusoids)
with $A = \begin{bmatrix} a_0 & a_1 & \dots & a_{K-1} \end{bmatrix}$

where $a_i = \begin{bmatrix} e^{-j\omega_i M} & \dots & 1 & \dots & e^{j\omega_i M} \end{bmatrix}^T$

we can write the Toeplitz matrix as

$$T = A \cdot \begin{bmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \dots & \\ & & & \alpha_{K-1} \end{bmatrix} \cdot A^M + N$$

where N is the noise Toeplitz matrix

Thus: If the sinusoids dominate the noise (M large enough), a K-dimensional subspace idendifies the sinusoids

Then:

1. **Compute SVD of T**
2. **Approximate by K largest singular value:** $T \to \hat{T}$
3. **Solve** $\hat{T}a = x$ **on subspace**
4. **Find roots closest to U.C.**

$\Rightarrow$ **best approximation of sinusoids**

Note:

- Many alternative available
- well studied problem
- time versus correlation domain

Example: - MUSIC
- ESPRIT
- NL

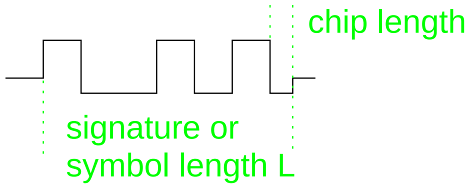
7.2 Multiuser Communication

Direct Sequence Code Division Mult. Access (DS-CDMA)

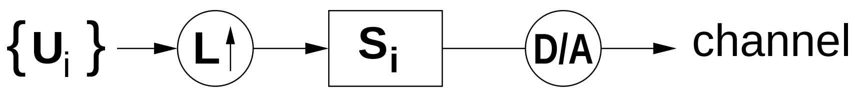
Model:

- User i has a signature sequence S_i
- each bit is spread into this signature

Ex:



Thus:



Clearly: rate of innovation is symbol rate

Usually: sampling done at chip rate or faster

Now: chip rate 10^2-10^3> symbole rate! (e.g. L=511)

But: - multiaccess scheme
- multipath environment

Multiaccess: signature are orthogonal
Multipath: small number of dominant pulses

User i: $p_i(t) = \sum_{k=1}^{p} \beta_i \delta(t - t_k^{(l)})$

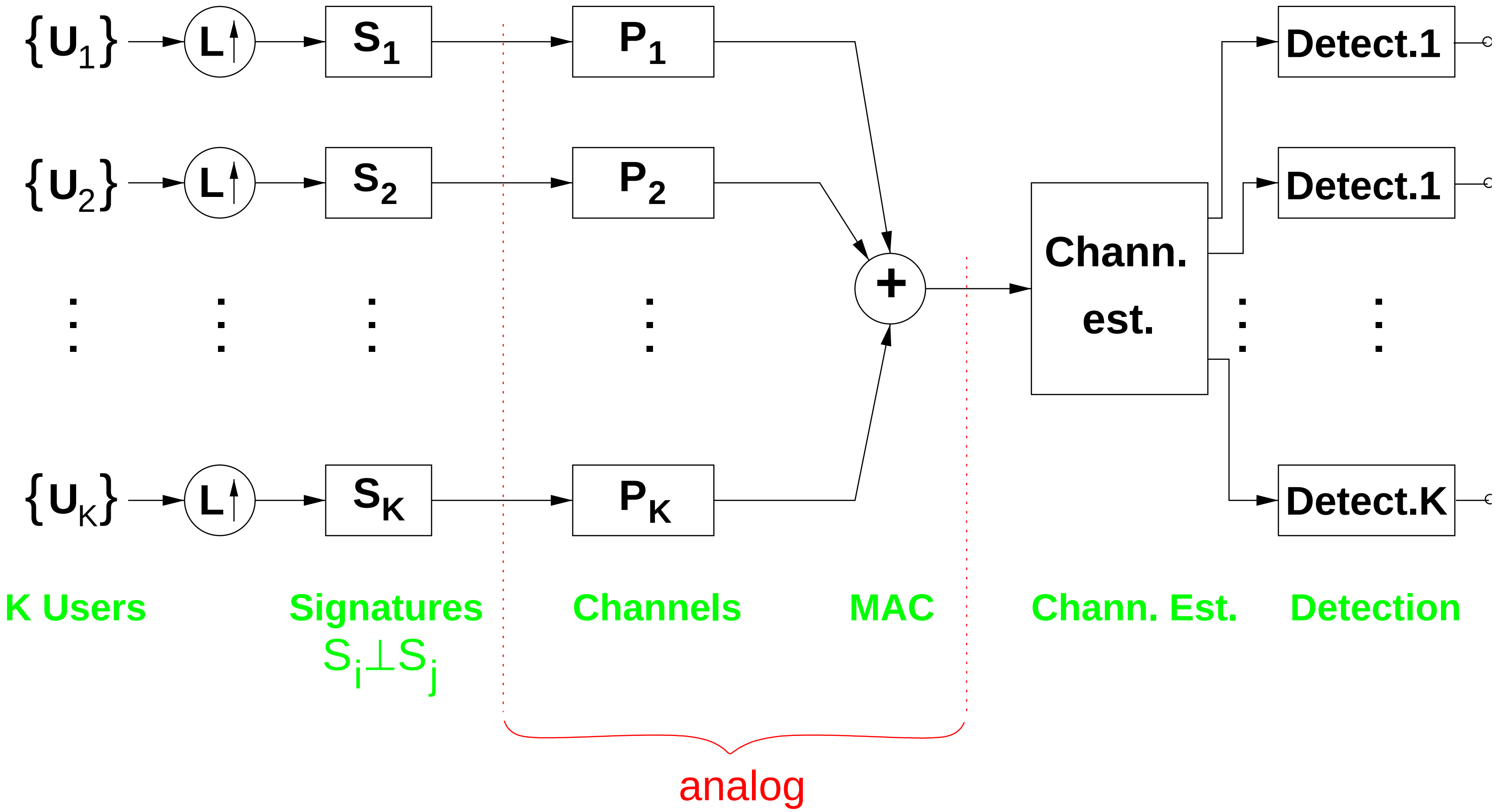
Two phases

1. **Channel estimation:**
   Using training sequences, $\{p_i(l)\}_{i = 1..K}$ is estimated

2. **Detection:**
   Based on the channel estimate, various detectors (e.g. MMSE) can be applied

Question: For a digital receiver,
Should one run:
- channel estimation
- detection

at symbol rate or chip rate?


{U1}
{U2}
{UK}
L↑
S1
S2
SK
P1
P2
PK
+
Chann. est.
Detect.1
Detect.1
Detect.K
K Users
Signatures
Si⊥Sj
Channels
MAC
Chann. Est.
Detection
analog

Degrees of freedom

Channels:

- **K users**
- **P multiple paths**

But: users can use training sequences of length K

Result:

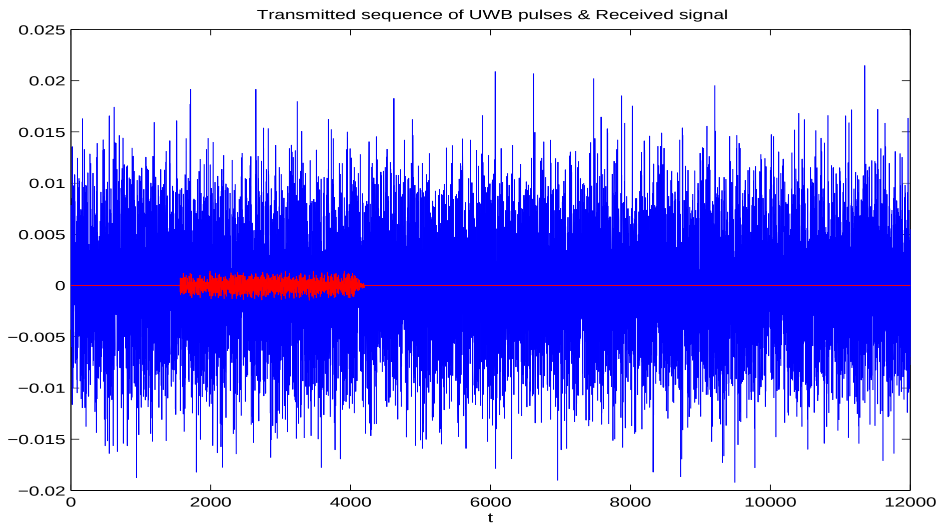
Solving K linear systems of O(M) with $M \geq 2P$**,**
is sufficient for channel estimation

7.3 Ultrawideband communications

Very low signal to noise ratio (-15 dB)

Used for communications in unlicensed spectrum and for ranging applications

Bandwidth: several GHz
Very difficult to design digital receivers



Results:

Finding one dominant eigenvalue can be sufficient !

8 Conclusions

We have seen:

- Many signals that look "unsampleable" actually can be sampled at their rate of innovation!
- Methods: give me an exponential and I will annihilate it!
- Structured linear systems with fast algorithms $O(K^2)$
- Can be generalized (rotational, 2D)

But: There are many more signals with finite rate of innovation

Conjecture: They can be sampled at or above their rate of innovation!

Outlook

- Many other parametric classes are of interest (piecewise trigonom.)
- Often, there is a “low degree of freedom” explanation
- This is not necessarily a subspace (e.g. manifold)
- “Super-resolution” signal processing for appropriate models (channels, images, etc...) has great potential

Occam’s Razor for sampling!

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