EE123
Digital Signal Processing

Lecture 6
Properties of DFT

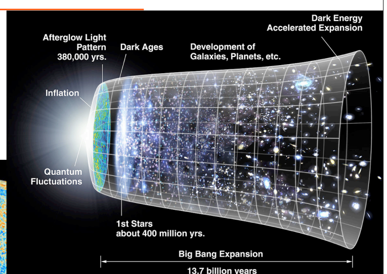
based on slides by J.M. Kahn

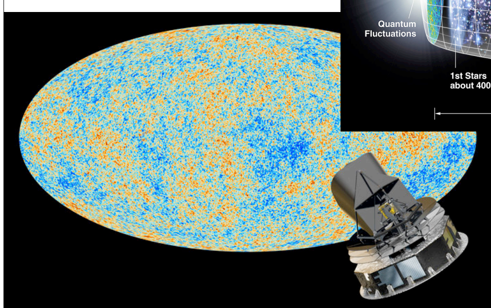
Announcements

- HW1 solutions posted -- self grading due
- HW2 due Friday
- SDR give after GSI Wednesday
- Finish reading Ch. 8, start Ch. 9
- ham radio licensing lectures Tue 6:30-8pm Cory 521

Cool things DSP

- Cosmic Microwave Background radiation





Last Time

- Discrete Fourier Transform
  - Similar to DFS
  - Sampling of the DTFT (subtitles....more later)
  - Properties of the DFT
- Today
  - Linear convolution with DFT
  - Fast Fourier Transform

Properties of DFT

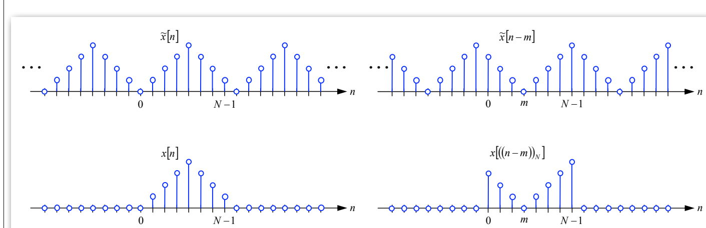
- Inherited from DFS (EE120/20) so no need to be proved
- Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

$$x[((n-m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

Circular shift



Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Show....

Properties of DFT

- Parseval's Identity

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

- Proof (in matrix notation)

$$\mathbf{x}^*\mathbf{x} = \left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right)^* \left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right) = \frac{1}{N^2}\mathbf{X}^*\underbrace{\mathbf{W}_N\mathbf{W}_N^*}_{N\cdot\mathbf{I}}\mathbf{X} = \frac{1}{N}\mathbf{X}^*\mathbf{X}$$

Circular Convolution Sum

- Circular Convolution:

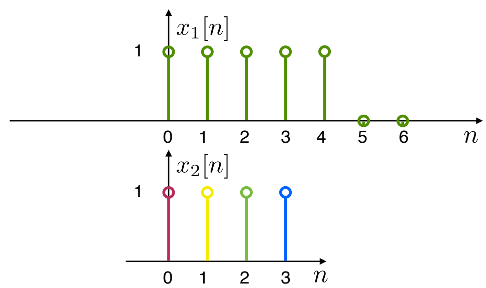
$$x_1[n] \; \textcircled{N} \; x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

for two signals of length N

- Note: Circular convolution is commutative

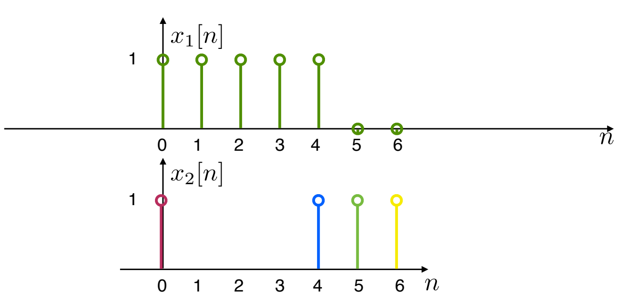
$$x_2[n] \; \textcircled{N} \; x_1[n] = x_1[n] \; \textcircled{N} \; x_2[n]$$

Compute Circular Convolution Sum



$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

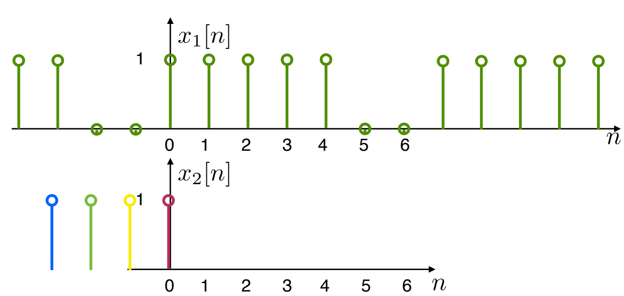
Compute Circular Convolution Sum



Circular 'flip'
multiply and add
Here: y[0]

$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

Compute Circular Convolution Sum

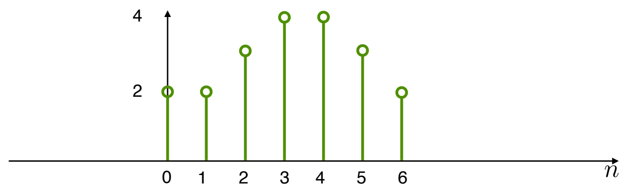


Equivalent periodic convolution over a period

$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

Result

$$y[n] = x_1[n] \textcircled{7} x_2[n] = ?$$



Properties of DFT

- Circular Convolution: Let x1[n], x2[n] be length N

$$x_1[n] \textcircled{N} x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

Very useful!!! (for linear convolutions with DFT)

- Multiplication: Let x1[n], x2[n] be length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \textcircled{N} X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Used DFT to do linear convolution

Linear Convolution

- We start with two non-periodic sequences:

$$\begin{aligned} &x[n] \quad 0 \leq n \leq L-1 \\ &h[n] \quad 0 \leq n \leq P-1 \end{aligned}$$

for example x[n] is a signal and h[n] an impulse response of a filter

- We want to compute the linear convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m]h[n-m]$$

y[n] is nonzero for 0 ≤ n ≤ L+P-2 with length **M=L+P-1**

- Requires L · P multiplications

Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

$$x_{\text{zp}}[n] = \begin{cases} x[n] & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq L+P-2 \end{cases}$$

- Zero-pad h[n] by L-1 zeros

$$h_{\text{zp}}[n] = \begin{cases} h[n] & 0 \leq n \leq P-1 \\ 0 & P \leq n \leq L+P-2 \end{cases}$$

- Now, both sequences are of length M=L+P-1

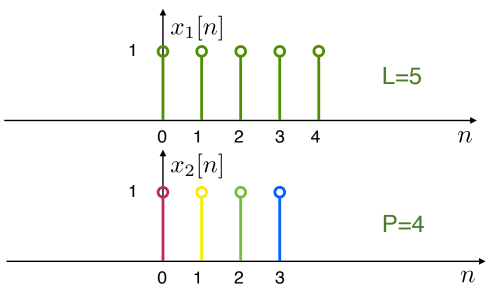
Linear Convolution via Circular Convolution

- Now, both sequences are of length M=L+P-1
- We can now compute the linear convolution using a circular one with length M = L+P-1

Linear convolution via circular

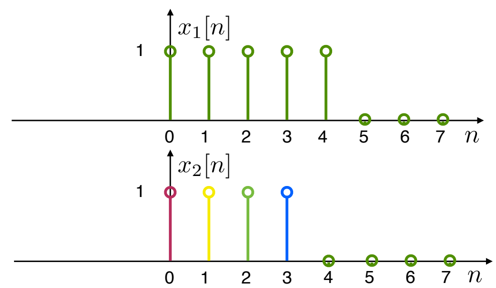
$$y[n] = x[n] * y[n] = \begin{cases} x_{\text{zp}}[n] \textcircled{M} h_{\text{zp}}[n] & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Example



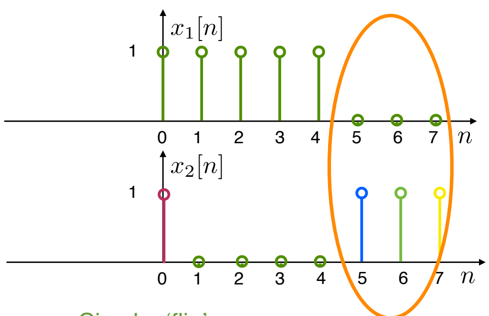
M = L + P - 1 = 8

Example



M = L + P - 1 = 8

Example



Circular 'flip'

M = L + P - 1 = 8

$$y[n] = x_1[n] \circledast_8 x_2[n] = x_1[n] * x_2[n]$$

Linear Convolution using DFT

- In practice we can implement a circulant convolution using the DFT property:

$$\begin{aligned} x[n] * h[n] &= x_{zp}[n] \circledast_M h_{zp}[n] \\ &= \mathcal{DFT}^{-1}\{\mathcal{DFT}\{x_{zp}[n]\} \cdot \mathcal{DFT}\{h_{zp}[n]\}\} \end{aligned}$$

for $0 \le n \le M-1$, $M=L+P-1$

- Advantage: DFT can be computed with $N\log_2 N$ complexity (FFT algorithm later!)
- Drawback: Must wait for all the samples -- huge delay -- incompatible with real-time

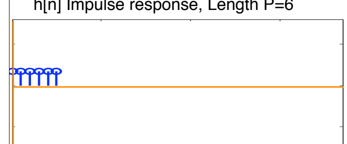
Block Convolution

- Problem:
  - An input signal x[n], has very long length (could be considered infinite)
  - An impulse response h[n] has length P
  - We want to take advantage of DFT/FFT and compute convolutions in blocks that are shorter than the signal
- Approach:
  - Break the signal into small blocks
  - Compute convolutions
  - Combine the results

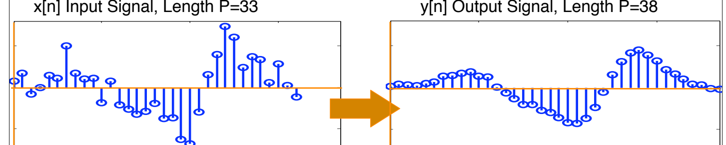
Block Convolution

Example:

h[n] Impulse response, Length P=6



x[n] Input Signal, Length P=33 y[n] Output Signal, Length P=38



Overlap-Add Method

We decompose the input signal $x[n]$ into non-overlapping segments $x_r[n]$ of length L:

$$x_r[n] = \begin{cases} x[n] & rL \le n \le (r+1)L - 1 \\ 0 & \text{otherwise} \end{cases}$$

The input signal is the sum of these input segments:

$$x[n] = \sum_{r=0}^{\infty} x_r[n]$$

The output signal is the sum of the output segments $x_r[n] * h[n]$:

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} x_r[n] * h[n] \qquad (1)$$

Each of the output segments $x_r[n] * h[n]$ is of length $N = L + P - 1$.

Overlap-Add Method

We can compute each output segment $x_r[n] * h[n]$ with linear convolution.

DFT-based circular convolution is usually more efficient:

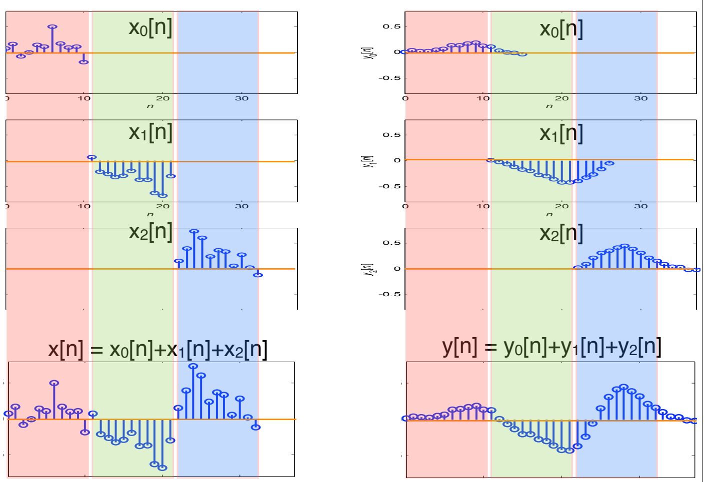
- Zero-pad input segment $x_r[n]$ to obtain $x_{r,zp}[n]$, of length N.
- Zero-pad the impulse response $h[n]$ to obtain $h_{zp}[n]$, of length N (this needs to be done only once).
- Compute each output segment using:

$$x_r[n] * h[n] = \mathcal{DFT}^{-1}\{\mathcal{DFT}\{x_{r,zp}[n]\} \cdot \mathcal{DFT}\{h_{zp}[n]\}\}$$

Since output segment $x_r[n] * h[n]$ starts offset from its neighbor $x_{r-1}[n] * h[n]$ by L, neighboring output segments overlap at $P - 1$ points.

Finally, we just add up the output segments using (1) to obtain the output.

Example of overlap and add:



Overlap-Save Method

Basic Idea

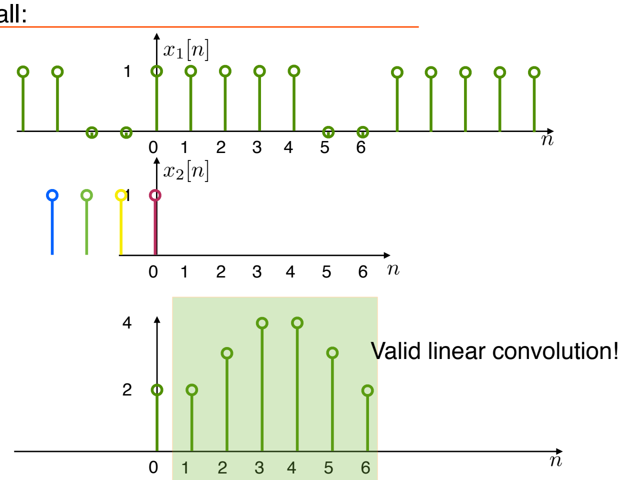
We split the input signal $x[n]$ into overlapping segments $x_r[n]$ of length $L + P - 1$.

Perform a circular convolution of each input segment $x_r[n]$ with the impulse response $h[n]$, which is of length P using the DFT.

Identify the L-sample portion of each circular convolution that corresponds to a linear convolution, and save it.

This is illustrated below where we have a block of L samples circularly convolved with a P sample filter.

Recall:



Example of overlap and save: