EE123 Digital Signal Processing

Lecture 8
FFT
Spectral Analysis

based on slides by J.M. Kahn

Announcements

- Last time:
  - Started FFT
- Today
  - Finish FFT
  - Start Frequency Analysis with DFT
- Read Ch. 10.1-10.2

- Who started playing with the SDR?

- Most FFT algorithms exploit the following properties of W_N^{kn}:
  - Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

  - Periodicity in n and k:

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

  - Power:

$$W_N^2 = W_{N/2}$$

- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - *Decimation-in-time* algorithms decompose $x[n]$ into successively smaller subsequences.
  - *Decimation-in-frequency* algorithms decompose $X[k]$ into successively smaller subsequences.
- We mostly discuss decimation-in-time algorithms here.

Assume length of $x[n]$ is power of 2 ($N = 2^\nu$). If smaller zero-pad to closest power.

Decimation-in-Time Fast Fourier Transform

- We start with the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, \ldots, N-1$$

- Separate the sum into even and odd terms:

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

These are two DFT's, each with half of the samples.

Decimation-in-Time Fast Fourier Transform

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk} \end{aligned}$$

- Note that:

$$W_N^{2rk} = e^{-j\left(\frac{2\pi}{N}\right)(2rk)} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}$$

Remember this trick, it will turn up often.

Decimation-in-Time Fast Fourier Transform

- Hence:

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk}$$
$$\triangleq G[k] + W_N^k H[k], \quad k = 0, \ldots, N-1$$

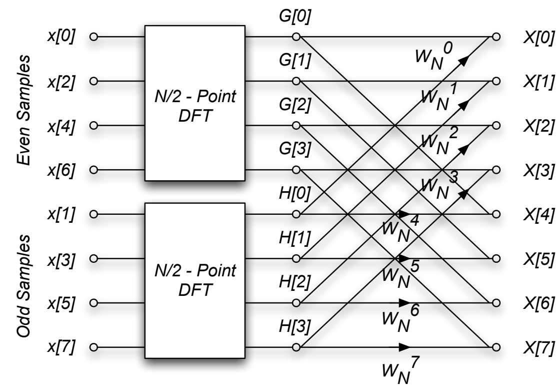
where we have defined:

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} \quad \Rightarrow \text{DFT of even idx}$$

$$H[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk} \quad \Rightarrow \text{DFT of odd idx}$$

Decimation-in-Time Fast Fourier Transform

An 8 sample DFT can then be diagrammed as



Decimation-in-Time Fast Fourier Transform

- Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$G[k+N/2] = \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{r(k+N/2)}$$
$$= \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk}W_{N/2}^{r(N/2)}$$
$$= \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk}$$
$$= G[k]$$

so

$$G[k+(N/2)] = G[k]$$
$$H[k+(N/2)] = H[k]$$

Decimation-in-Time Fast Fourier Transform

- The periodicity of $G[k]$ and $H[k]$ allows us to further simplify.
- For the first $N/2$ points we calculate $G[k]$ and $W_N^k H[k]$, and then compute the sum

$$X[k] = G[k] + W_N^k H[k] \qquad \forall\{k : 0 \le k < \frac{N}{2}\}.$$

How does periodicity help for $\frac{N}{2} \le k < N$?

Decimation-in-Time Fast Fourier Transform

$$X[k] = G[k] + W_N^k H[k] \qquad \forall\{k : 0 \le k < \frac{N}{2}\}.$$

- for $\frac{N}{2} \le k < N$:

$$W_N^{k+(N/2)} = ?$$

$$X[k+(N/2)] = ?$$

Decimation-in-Time Fast Fourier Transform

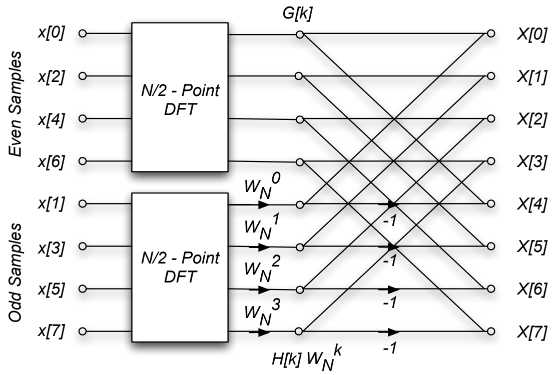
$$X[k+(N/2)] = G[k] - W_N^k H[k]$$

We previously calculated $G[k]$ and $W_N^k H[k]$.

Now we only have to compute their difference to obtain the second half of the spectrum. No additional multiplies are required.

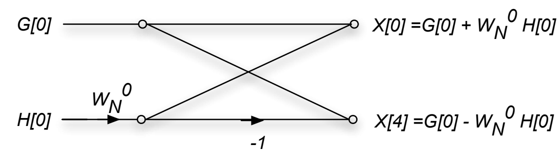
Decimation-in-Time Fast Fourier Transform

- The N-point DFT has been reduced two $N/2$-point DFTs, plus $N/2$ complex multiplications. The 8 sample DFT is then:



Decimation-in-Time Fast Fourier Transform

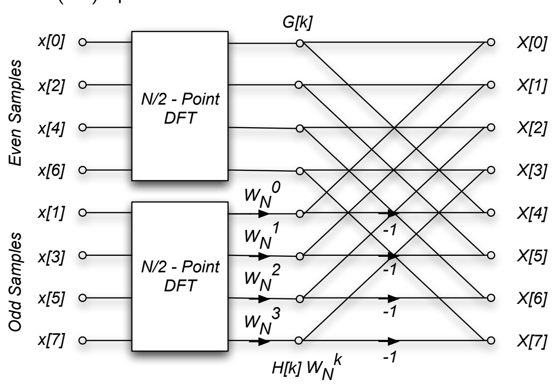
- Note that the inputs have been reordered so that the outputs come out in their proper sequence.
- We can define a *butterfly operation*, e.g., the computation of $X[0]$ and $X[4]$ from $G[0]$ and $H[0]$:



This is an important operation in DSP.

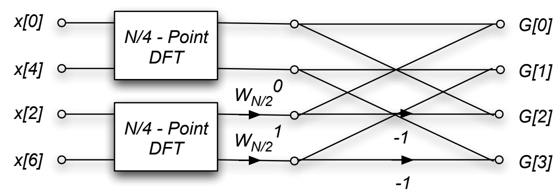
Decimation-in-Time Fast Fourier Transform

- Still $O(N^2)$ operations..... What shall we do?



Decimation-in-Time Fast Fourier Transform

- We can use the same approach for each of the $N/2$ point DFT's. For the $N = 8$ case, the $N/2$ DFTs look like



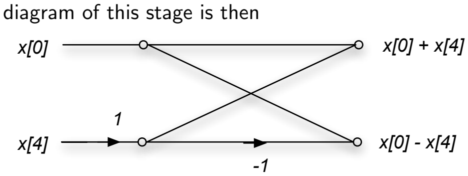
*Note that the inputs have been reordered again.

Decimation-in-Time Fast Fourier Transform

- At this point for the 8 sample DFT, we can replace the $N/4 = 2$ sample DFT's with a single butterfly.
  The coefficient is

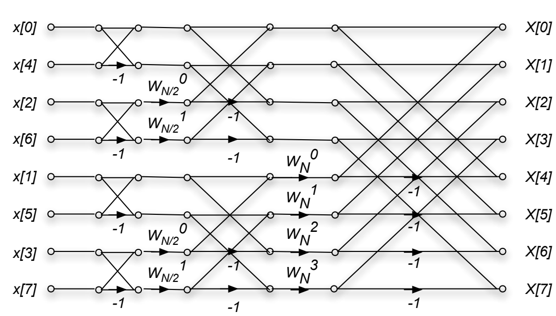
$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$

  The diagram of this stage is then



Decimation-in-Time Fast Fourier Transform

Combining all these stages, the diagram for the 8 sample DFT is:



This the decimation-in-time FFT algorithm.

Decimation-in-Time Fast Fourier Transform

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires $N/2$ complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2)\log_2 N$.
- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even. Zero low-order bit first
  - Next stage repeats with next zero-lower bit first.
  - Net effect is reversing the bit order of indexes

Decimation-in-Time Fast Fourier Transform

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary	Bit-Reversed Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

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Decimation-in-Frequency Fast Fourier Transform

The DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

If we only look at the even samples of $X[k]$, we can write $k = 2r$,

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}$$

We split this into two sums, one over the first $N/2$ samples, and the second of the last $N/2$ samples.

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)}$$

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Decimation-in-Frequency Fast Fourier Transform

But $W_N^{2r(n+N/2)} = W_N^{2rn} W_N^N = W_N^{2rn} = W_{N/2}^{rn}$.
We can then write

$$\begin{aligned} X[2r] &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)} \\ &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2rn} \\ &= \sum_{n=0}^{(N/2)-1} (x[n] + x[n+N/2])\, W_{N/2}^{rn} \end{aligned}$$

This is the $N/2$-length DFT of first and second half of $x[n]$ summed.

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Decimation-in-Frequency Fast Fourier Transform

$$\begin{aligned} X[2r] &= \text{DFT}_{\frac{N}{2}} \{(x[n] + x[n+N/2])\} \\ X[2r+1] &= \text{DFT}_{\frac{N}{2}} \{(x[n] - x[n+N/2])\, W_N^n\} \end{aligned}$$

(By a similar argument that gives the odd samples)

Continue the same approach is applied for the $N/2$ DFTs, and the $N/4$ DFT's until we reach simple butterflies.

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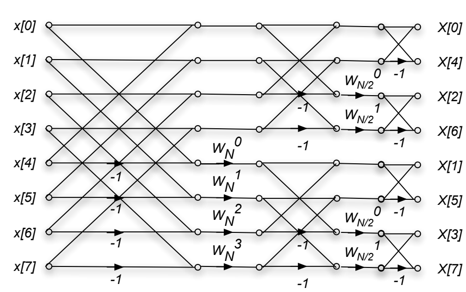
Decimation-in-Frequency Fast Fourier Transform

The diagram for and 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

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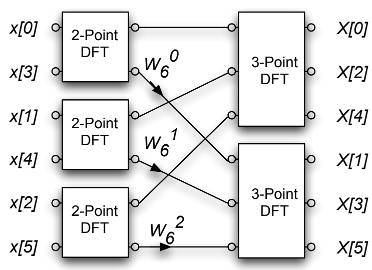
Non-Power-of-2 FFT's

A similar argument applies for any length DFT, where the length N is a composite number.
For example, if $N = 6$, a decimation-in-time FFT could compute three 2-point DFT's followed by two 3-point DFT's



Non-Power-of-2 FFT's

Good component DFT's are available for lengths up to 20 or so.
Many of these exploit the structure for that specific length. For example, a factor of

$$W_N^{N/4} = e^{-j\frac{2\pi}{N}(N/4)} = e^{-j\frac{\pi}{2}} = -j \qquad \text{Why?}$$

just swaps the real and imaginary components of a complex number, and doesn't actually require any multiplies.
Hence a DFT of length 4 doesn't require any complex multiplies.
Half of the multiplies of an 8-point DFT also don't require multiplication.
Composite length FFT's can be very efficient for any length that factors into terms of this order.

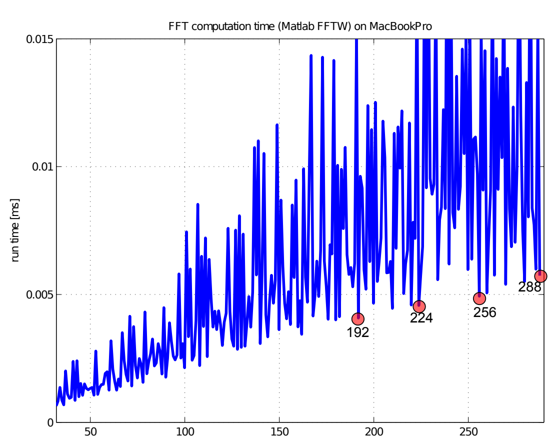
For example $N = 693$ factors into

$$N = (7)(9)(11)$$

each of which can be implemented efficiently. We would perform

- 9×11 DFT's of length 7
- 7×11 DFT's of length 9, and
- 7×9 DFT's of length 11

- Historically, the power-of-two FFTs were much faster (better written and implemented).
- For non-power-of-two length, it was faster to zero pad to power of two.
- Recently this has changed. The free FFTW package implements very efficient algorithms for almost any filter length. Matlab has used FFTW since version 6



FFT as Matrix Operation

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- W_N is fully populated $\Rightarrow N^2$ entries.

FFT as Matrix Operation

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- W_N is fully populated $\Rightarrow N^2$ entries.
- FFT is a decomposition of W_N into a more sparse form:

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} W_{N/2} & 0 \\ 0 & W_{N/2} \end{bmatrix} \begin{bmatrix} \text{Even-Odd Perm.} \\ \text{Matrix} \end{bmatrix}$$

- $I_{N/2}$ is an identity matrix. $D_{N/2}$ is a diagonal with entries $1,\ W_N,\ \cdots,\ W_N^{N/2-1}$

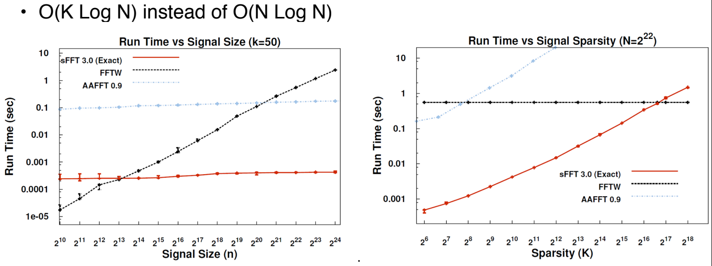
FFT as Matrix Operation

Example: $N = 4$

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & W_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -W_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beyond NlogN

- What if the signal x[n] has a k sparse frequency
  - A. Gilbert et. al, "Near-optimal sparse Fourier representations via sampling
  - H. Hassanieh et. al, "Nearly Optimal Sparse Fourier Transform"
  - Others......
- O(K Log N) instead of O(N Log N)



From: http://groups.csail.mit.edu/netmit/sFFT/paper.html

Spectral Analysis with the DFT

The DFT can be used to analyze the spectrum of a signal.

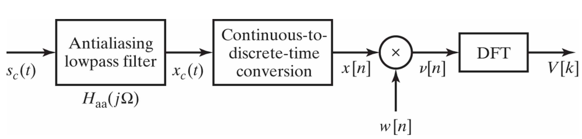
It would seem that this should be simple, take a block of the signal and compute the spectrum with the DFT.

However, there are many important issues and tradeoffs:

- Signal duration vs spectral resolution
- Signal sampling rate vs spectral range
- Spectral sampling rate
- Spectral artifacts

Spectral Analysis with the DFT

Consider these steps of processing continuous-time signals:



Spectral Analysis with the DFT

Two important tools:

- Applying a window to the input signal – reduces spectral artifacts
- Padding input signal with zeros – increases the spectral sampling

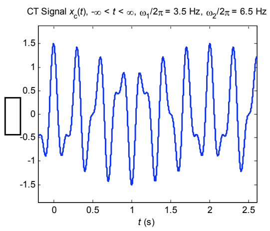
Key Parameters:

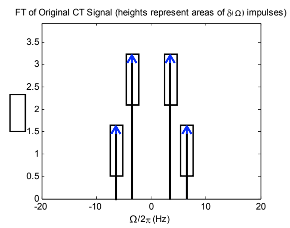
Parameter	Symbol	Units
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

Filtered Continuous-Time Signal

We consider an example:

$$x_c(t) = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$$
$$X_c(j\Omega) = A_1\pi[\delta(\Omega-\omega_1)+\delta(\Omega+\omega_1)] + A_2\pi[\delta(\Omega-\omega_2)+\delta(\Omega+\omega_2)]$$





Sampled Filtered Continuous-Time Signal

Sampled Signal
If we sampled the signal over an infinite time duration, we would have:

$$x[n] = x_c(t)|_{t=nT}, \quad -\infty < n < \infty$$

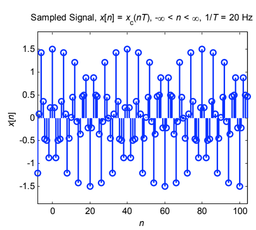
described by the discrete-time Fourier transform:

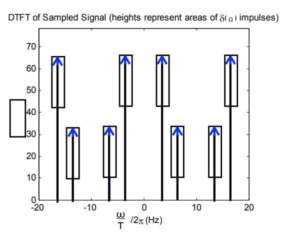
$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\Omega - r\frac{2\pi}{T}\right)\right), \quad -\infty < \Omega < \infty$$

Recall $X(e^{j\omega}) = X(e^{j\Omega T})$, where $\omega = \Omega T$ ... more in ch 4.

Sampled Filtered Continuous-Time Signal

In the examples shown here, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz, sufficiently high that aliasing does not occur.





Windowed Sampled Signal

Block of L Signal Samples
In any real system, we sample only over a finite block of L samples:

$$x[n] = x_c(t)|_{t=nT}, \quad 0 \le n \le L-1$$

This simply corresponds to a rectangular window of duration L.

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing

Windowed Sampled Signal

Windowed Block of L Signal Samples
We take the block of signal samples and multiply by a window of duration L, obtaining:

$$v[n] = x[n]\cdot w[n], \quad 0 \le n \le L-1$$

Suppose the window $w[n]$ has DTFT $W(e^{j\omega})$.

Then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

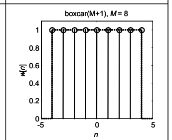
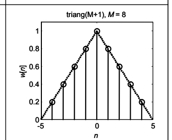
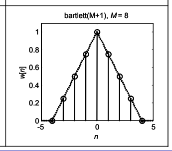
Windowed Sampled Signal

Convolution with $W(e^{j\omega})$ has two effects in the spectrum:

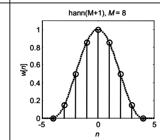
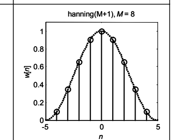
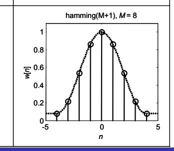
1. It limits the spectral resolution. – Main lobes of the DTFT of the window
2. The window can produce *spectral leakage*. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle

Windows (as defined in MATLAB)

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Rectangular Boxcar Fourier	$w[n] = \begin{cases} 1 & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`boxcar(M+1)`	
Triangular	$w[n] = \begin{cases} 1 - \dfrac{\lvert n\rvert}{M/2+1} & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`triang(M+1)`	
Bartlett	$w[n] = \begin{cases} 1 - \dfrac{\lvert n\rvert}{M/2} & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`bartlett(M+1)`	

Windows (as defined in MATLAB)

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \dfrac{1}{2}\left[1 + \cos\left(\dfrac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \dfrac{1}{2}\left[1 + \cos\left(\dfrac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Windows

- All of the window functions $w[n]$ are real and even.
- All of the discrete-time Fourier transforms

$$W(e^{j\omega}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n]e^{-jn\omega}$$

are real, even, and periodic in ω with period 2π.

- In the following plots, we have normalized the windows to unit d.c. gain:

$$W(e^{j0}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n] = 1$$

This makes it easier to compare windows.

Window Example