EE123
Digital Signal Processing

Lecture 9

based on slides by J.M. Kahn

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Announcements

- Last time:
  - FFT
- Today:
  - Frequency analysis with DFT
  - Windowing
  - Effect of zero-padding

Spectral Analysis with the DFT

The DFT can be used to analyze the spectrum of a signal.

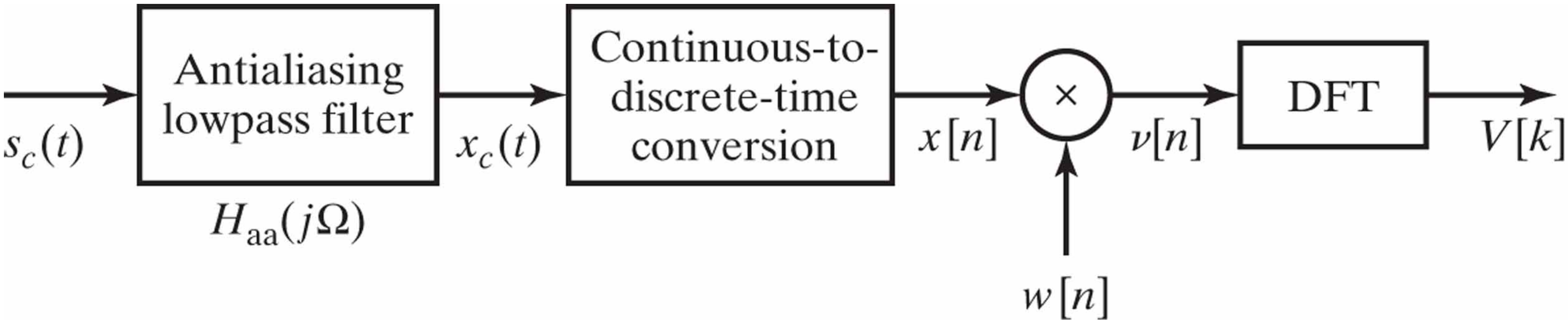
It would seem that this should be simple, take a block of the signal and compute the spectrum with the DFT.

However, there are many important issues and tradeoffs:

- Signal duration vs spectral resolution
- Signal sampling rate vs spectral range
- Spectral sampling rate
- Spectral artifacts

Spectral Analysis with the DFT

Consider these steps of processing continuous-time signals:



Spectral Analysis with the DFT

Two important tools:

- Applying a window to the input signal – reduces spectral artifacts
- Padding input signal with zeros – increases the spectral sampling

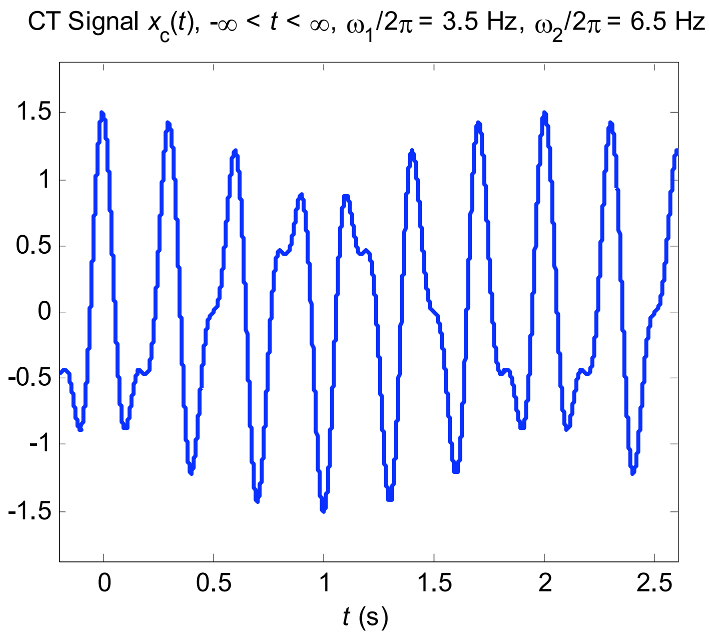
Key Parameters:

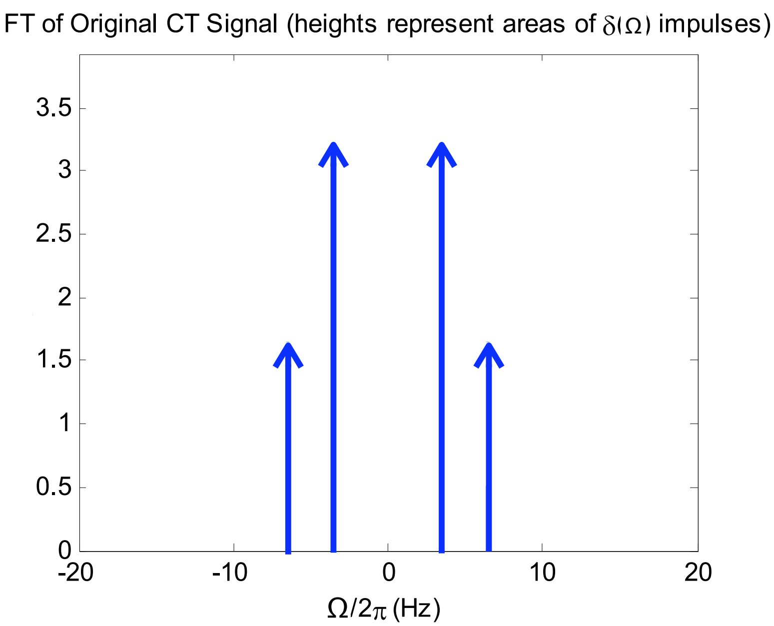
Parameter	**Symbol**	**Units**
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

Filtered Continuous-Time Signal

We consider an example:

$$
\begin{aligned}
x_c(t) &= A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \\
X_c(j\Omega) &= A_1 \pi[\delta(\Omega - \omega_1) + \delta(\Omega + \omega_1)] + A_2 \pi[\delta(\Omega - \omega_2) + \delta(\Omega + \omega_2)]
\end{aligned}
$$





Sampled Filtered Continuous-Time Signal

Sampled Signal

If we sampled the signal over an infinite time duration, we would have:

$$x[n] = x_c(t)|_{t=nT}, \qquad -\infty < n < \infty$$

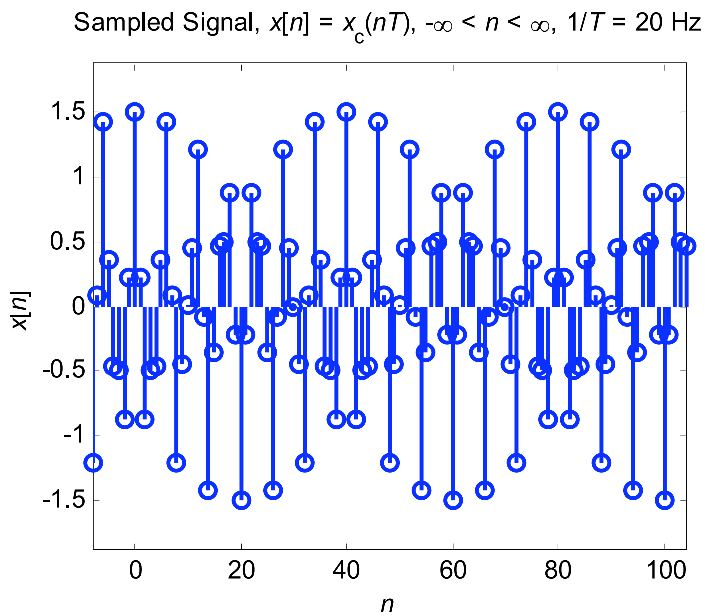
described by the discrete-time Fourier transform:

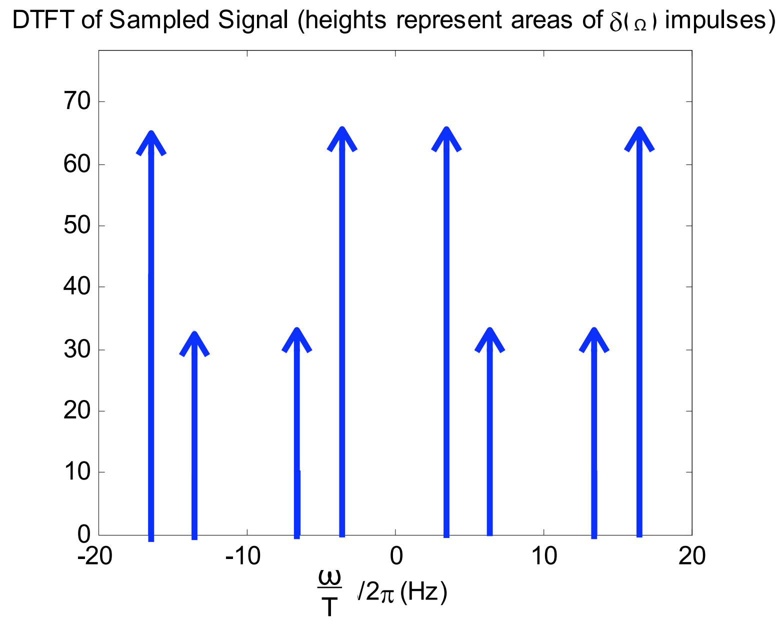
$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\Omega - r\frac{2\pi}{T}\right)\right), \qquad -\infty < \Omega < \infty$$

Recall $X(e^{j\omega}) = X(e^{j\Omega T})$, where $\omega = \Omega T$ ... more in ch 4.

Sampled Filtered Continuous-Time Signal

In the examples shown here, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz, sufficiently high that aliasing does not occur.





Windowed Sampled Signal

Block of L Signal Samples

In any real system, we sample only over a finite block of L samples:

$$x[n] = x_c(t)|_{t=nT}, \quad 0 \leq n \leq L-1$$

This simply corresponds to a rectangular window of duration L.

Recall: in Homework 1 we explored the effect of rectangular and triangular windowing

Windowed Sampled Signal

Windowed Block of L Signal Samples

We take the block of signal samples and multiply by a window of duration L, obtaining:

$$v[n] = x[n] \cdot w[n], \qquad 0 \leq n \leq L-1$$

Suppose the window $w[n]$ has DTFT $W(e^{j\omega})$.

Then the windowed block of signal samples has a DTFT given by the periodic convolution between $X(e^{j\omega})$ and $W(e^{j\omega})$:

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

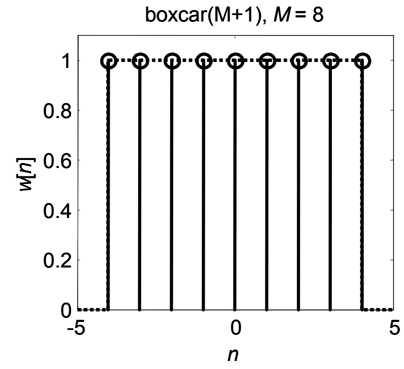
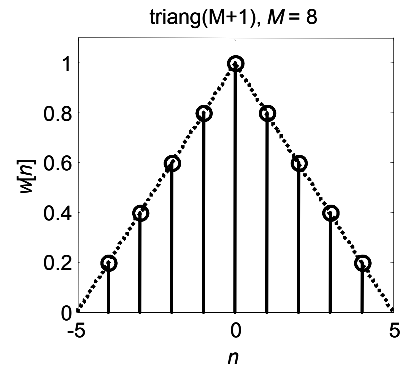
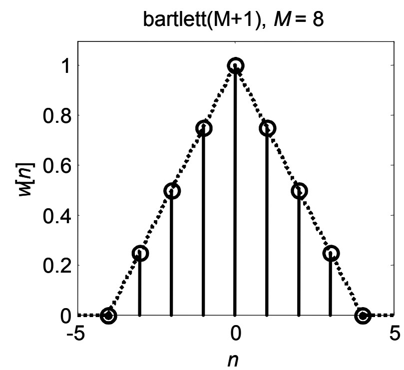
Windowed Sampled Signal

Convolution with $W(e^{j\omega})$ has two effects in the spectrum:

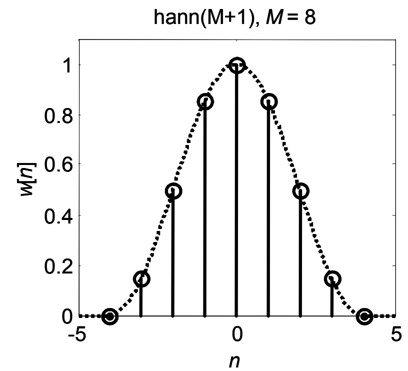
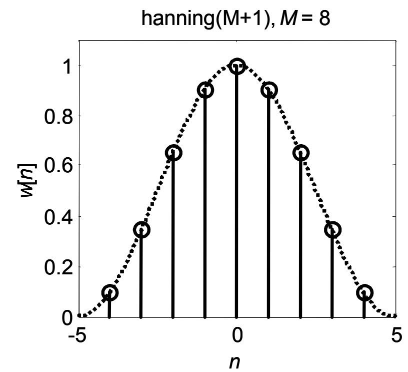
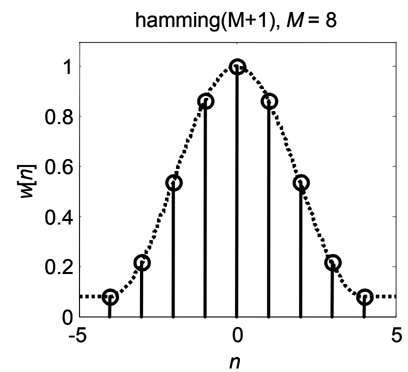
1. It limits the spectral resolution. – Main lobes of the DTFT of the window
2. The window can produce *spectral leakage*. – Side lobes of the DTFT of the window

* These two are always a tradeoff - time-frequency uncertainty principle

Windows (as defined in MATLAB)

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Rectangular Boxcar Fourier	$w[n]=\begin{cases}1 & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`boxcar(M+1)`	
Triangular	$w[n]=\begin{cases}1-\dfrac{\lvert n\rvert}{M/2+1} & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`triang(M+1)`	
Bartlett	$w[n]=\begin{cases}1-\dfrac{\lvert n\rvert}{M/2} & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2\end{cases}$	`bartlett(M+1)`	

Windows (as defined in MATLAB)

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Windows

- All of the window functions $w[n]$ are real and even.
- All of the discrete-time Fourier transforms

$$W(e^{j\omega}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n]e^{-jn\omega}$$

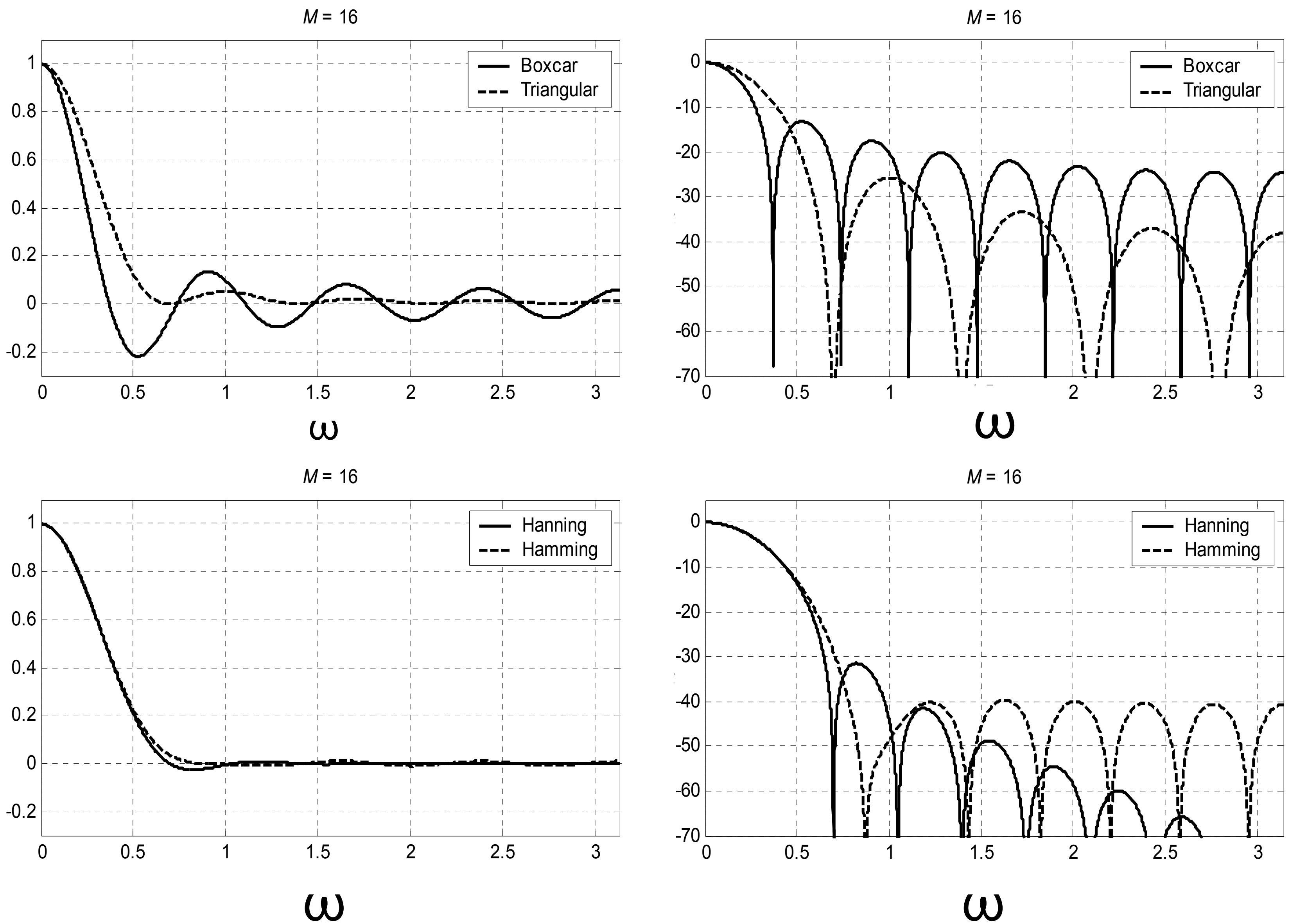
are real, even, and periodic in ω with period 2π.

- In the following plots, we have normalized the windows to unit d.c. gain:

$$W(e^{j0}) = \sum_{n=-\frac{M}{2}}^{\frac{M}{2}} w[n] = 1$$

This makes it easier to compare windows.

Window Example



Windows Properties

These are characteristic of the window type

Window	Main-lobe	Sidelobe δ_s	Sidelobe $-20\log_{10}\delta_s$
Rect	$\frac{4\pi}{M+1}$	0.09	21
Bartlett	$\frac{8\pi}{M+1}$	0.05	26
Hann	$\frac{8\pi}{M+1}$	0.0063	44
Hamming	$\frac{8\pi}{M+1}$	0.0022	53
Blackman	$\frac{12\pi}{M+1}$	0.0002	74

Most of these (Bartlett, Hann, Hamming) have a transition width that is twice that of the rect window.

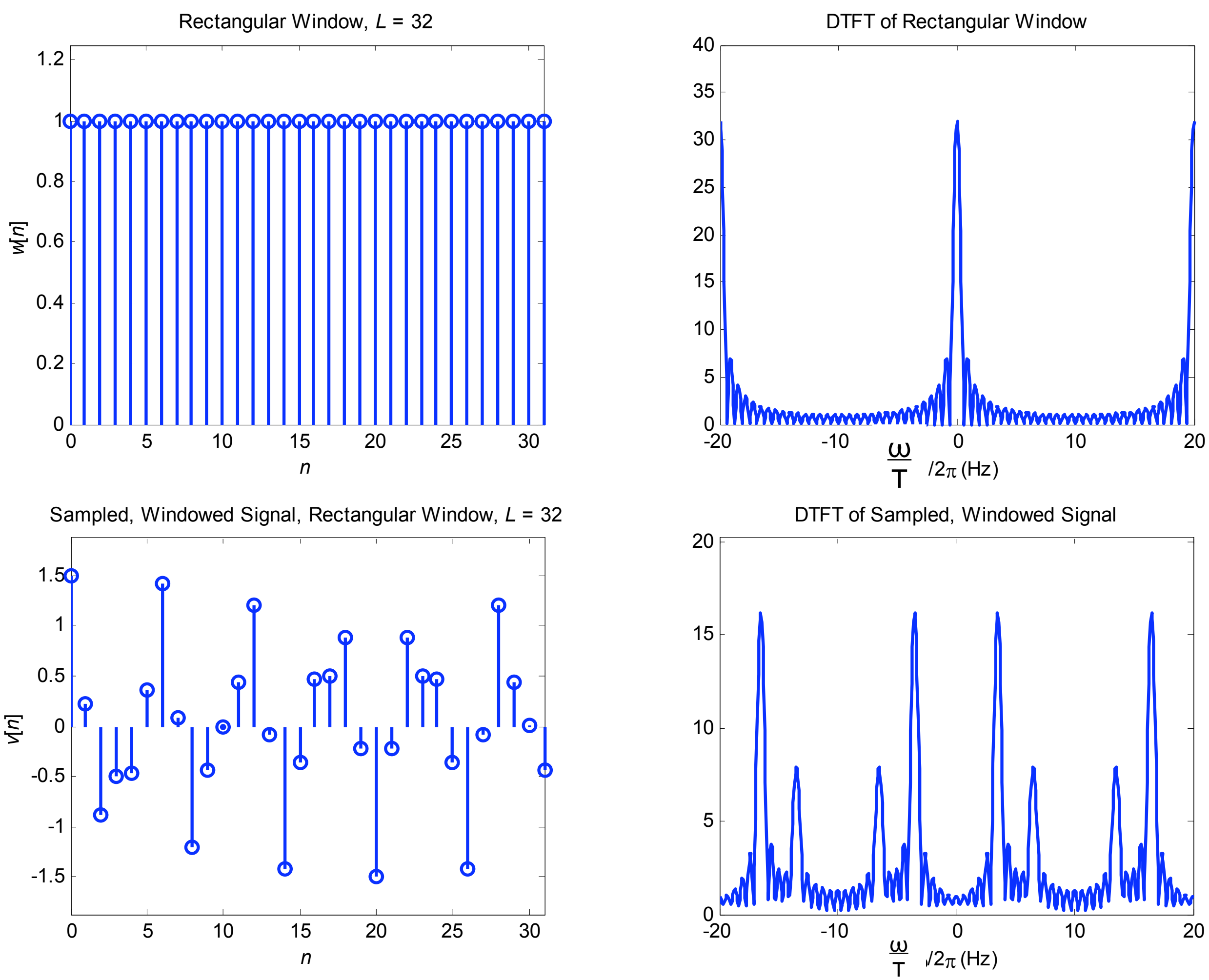
Warning: Always check what's the definition of M

Adapted from *A Course In Digital Signal Processing* by Boaz Porat, Wiley, 1997

Windows Examples

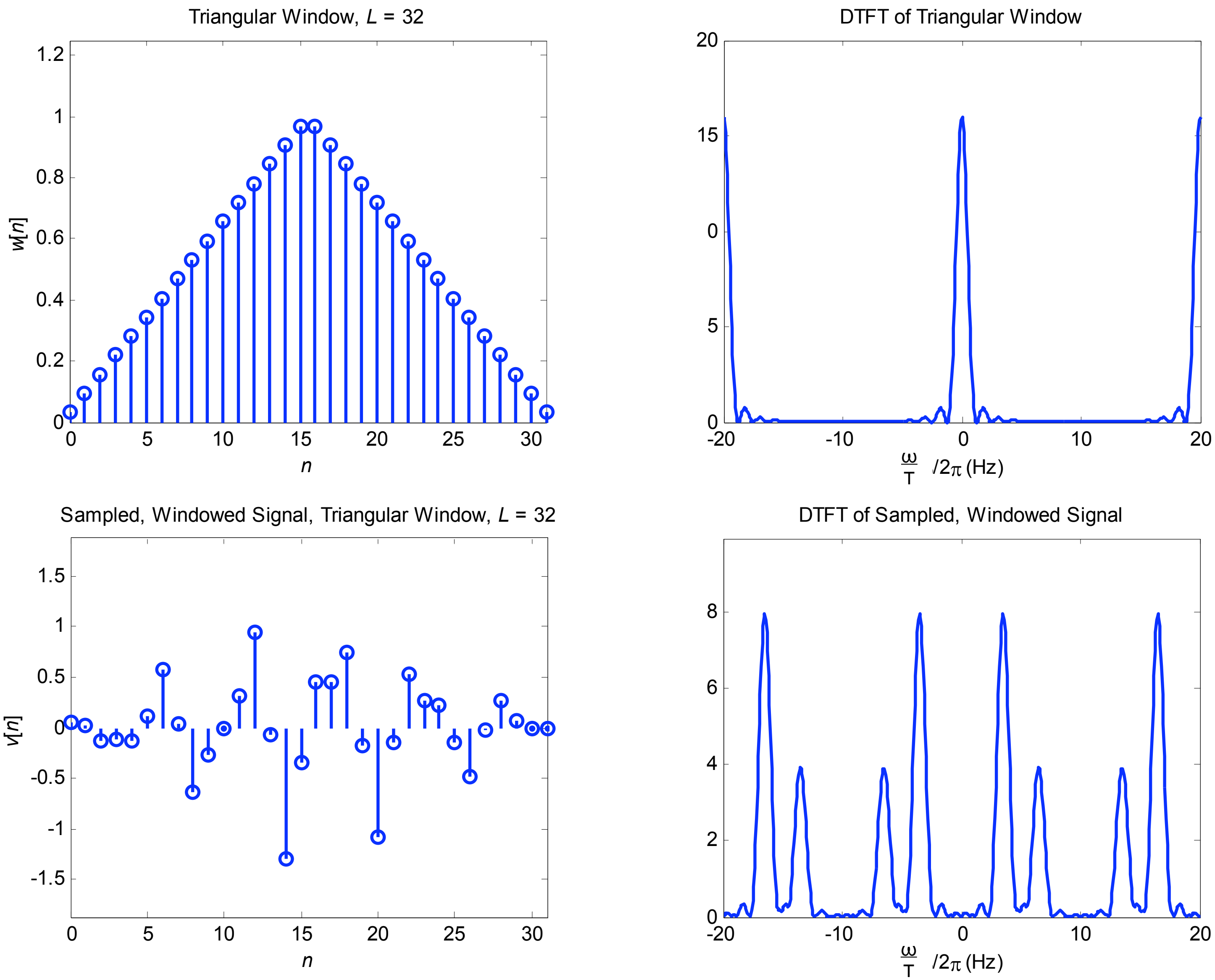
Here we consider several examples. As before, the sampling rate is $\Omega_s/2\pi = 1/T = 20$ Hz.

Rectangular Window, $L = 32$

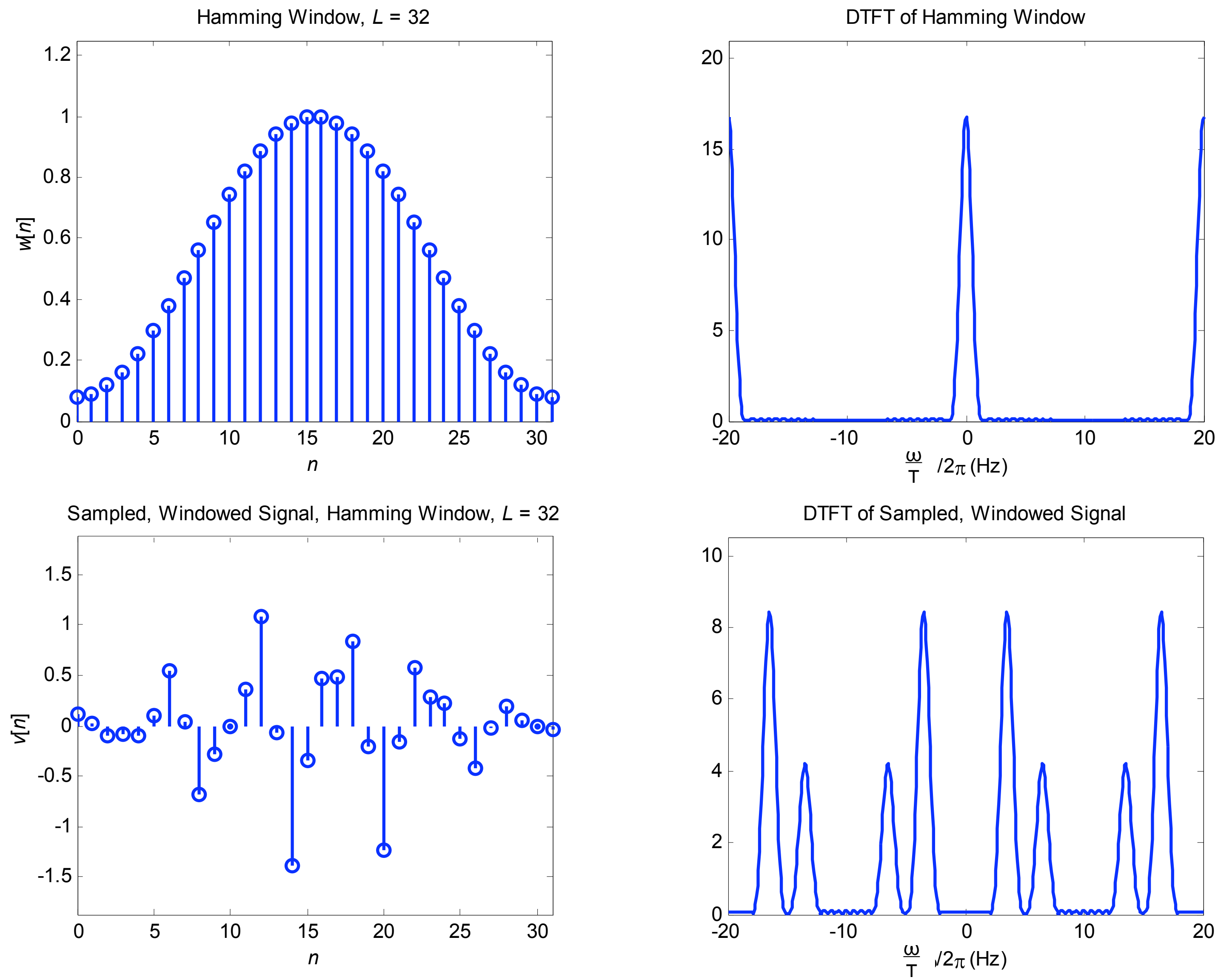


Windows Examples

Triangular Window, $L = 32$

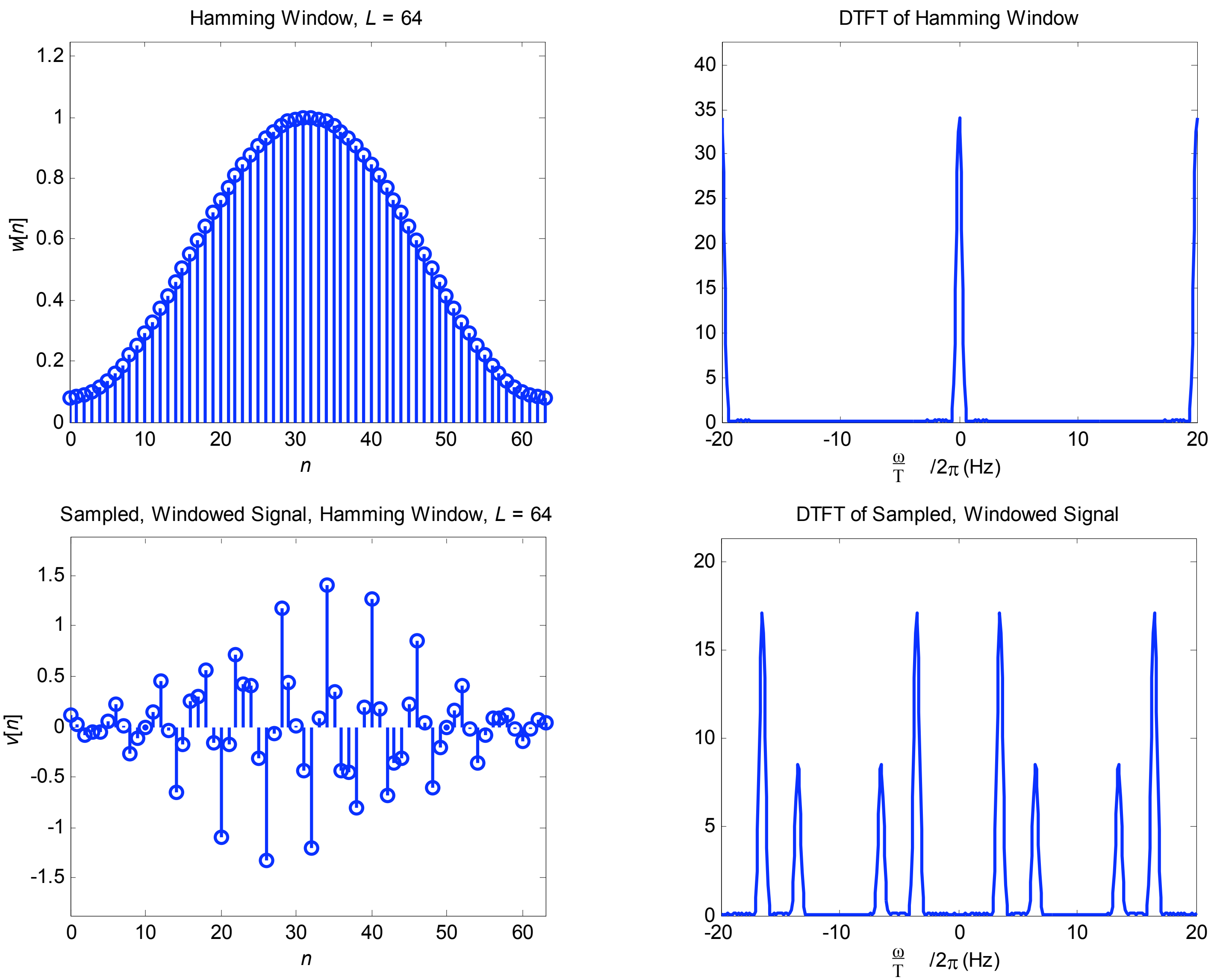


Hamming Window, $L = 32$



Windows Examples

Hamming Window, $L = 64$



Optimal Window: Kaiser

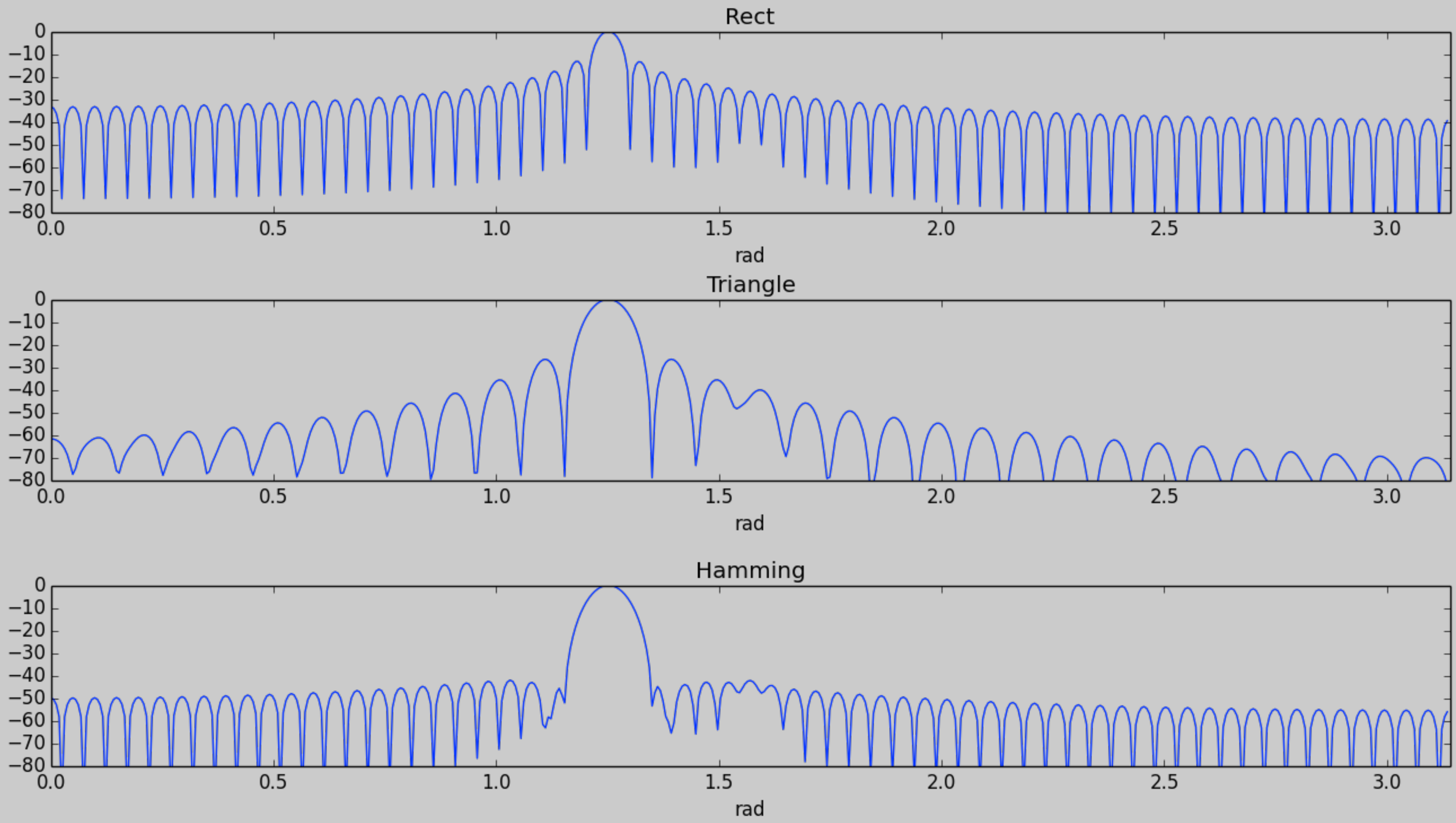
- Minimum main-lobe width for a given side-lobe energy %

$$\frac{\int_{\text{sidelobes}} |H(e^{j\omega})|^2 d\omega}{\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega}$$

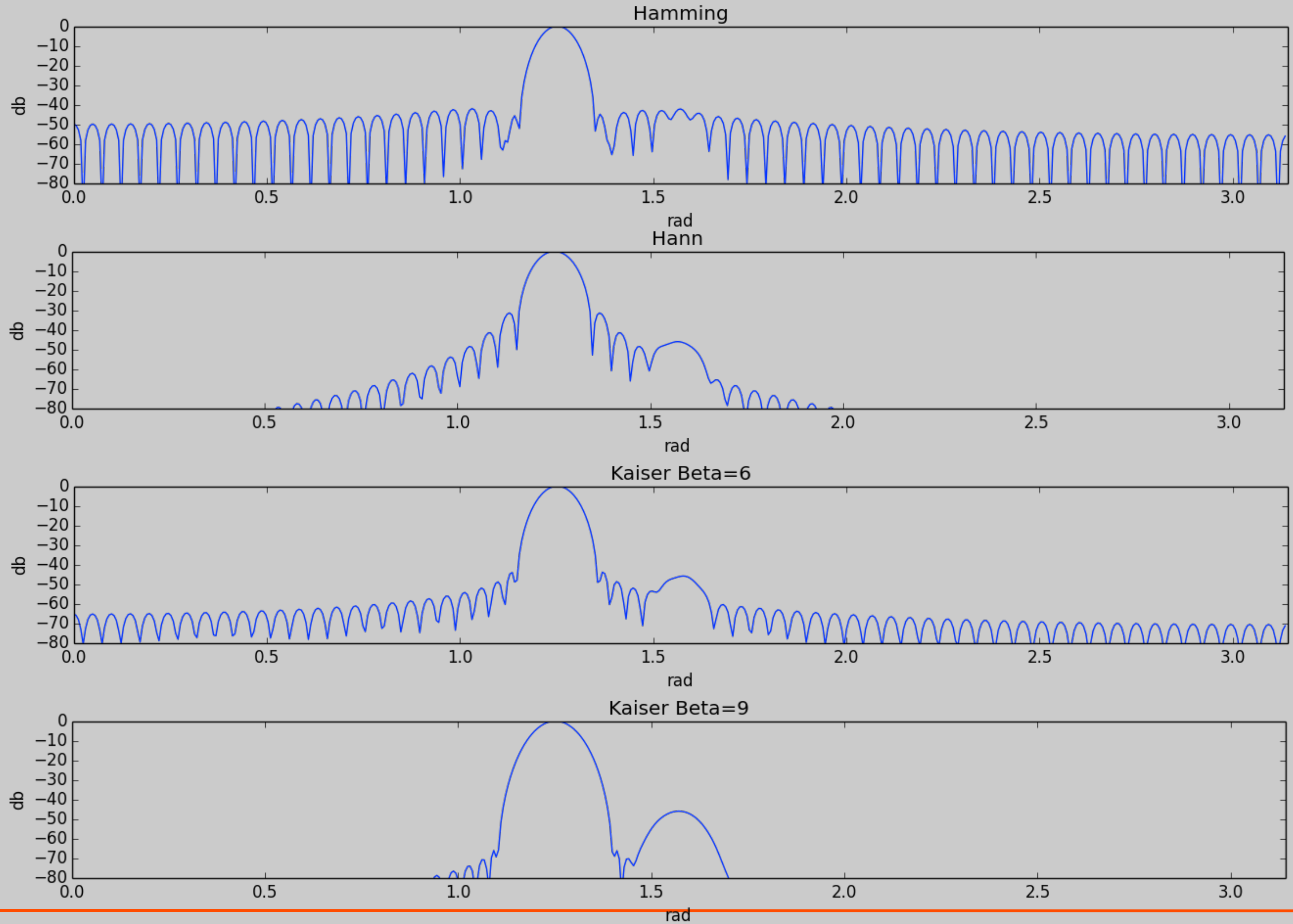
- Window is parametrized with L and β OS Eq 10.12
  - β determines side-lobe level
  - L determines main-lobe width

Example

$$y = \sin(2\pi 0.1992n) + 0.005\sin(2\pi 0.25n) \quad | \quad 0 \leq n < 128$$



Example


Hamming
db
0
−10
−20
−30
−40
−50
−60
−70
−80
0.0
0.5
1.0
1.5
2.0
2.5
3.0
rad
Hann
Kaiser Beta=6
Kaiser Beta=9

Zero-Padding

- In preparation for taking an N-point DFT, we may zero-pad the windowed block of signal samples to a block length $N \geq L$:

$$\begin{cases} v[n] & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq N-1 \end{cases}$$

- This zero-padding has no effect on the DTFT of $v[n]$, since the DTFT is computed by summing over $-\infty < n < \infty$.

Effect of Zero Padding

- We take the N-point DFT of the zero-padded $v[n]$, to obtain the block of N spectral samples:

$$V[k], \quad 0 \leq k \leq N-1$$

Zero-Padding

- Consider the DTFT of the zero-padded $v[n]$. Since the zero-padded $v[n]$ is of length N, its DTFT can be written:

$$V(e^{j\omega}) = \sum_{n=0}^{N-1} v[n]e^{-jn\omega}, \qquad -\infty < \omega < \infty$$

The N-point DFT of $v[n]$ is given by:

$$V[k] = \sum_{n=0}^{N-1} v[n]W_N^{kn} = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)nk}, \qquad 0 \leq k \leq N-1$$

We see that $V[k]$ corresponds to the samples of $V(e^{j\omega})$:

$$V[k] = \left. V(e^{j\omega})\right|_{\omega = k\frac{2\pi}{N}}, \qquad 0 \leq k \leq N-1$$

To obtain samples at more closely spaced frequencies, we zero-pad $v[n]$ to longer block length N. The spectrum is the same, we just have more samples.

Frequency Analysis with DFT

- Note that the ordering of the DFT samples is unusual.

$$V[k] = \sum_{n=0}^{N-1} v[n] W_N^{nk}$$

The DC sample of the DFT is $k = 0$

$$V[0] = \sum_{n=0}^{N-1} v[n] W_N^{0n} = \sum_{n=0}^{N-1} v[n]$$

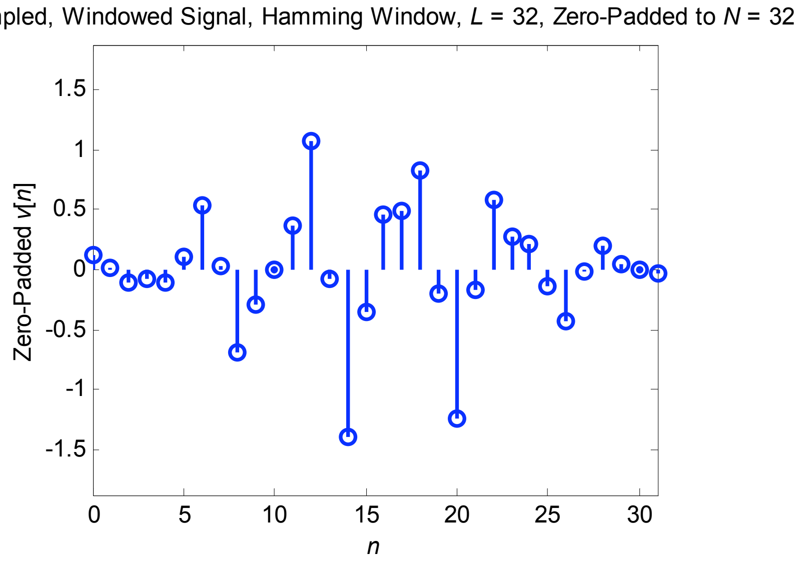
- The positive frequencies are the first $N/2$ samples
- The first $N/2$ negative frequencies are circularly shifted

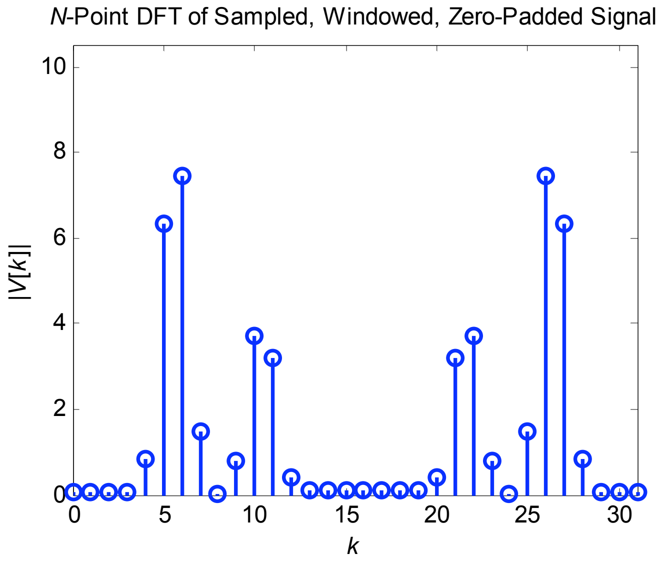
$$((-k))_N = N - k$$

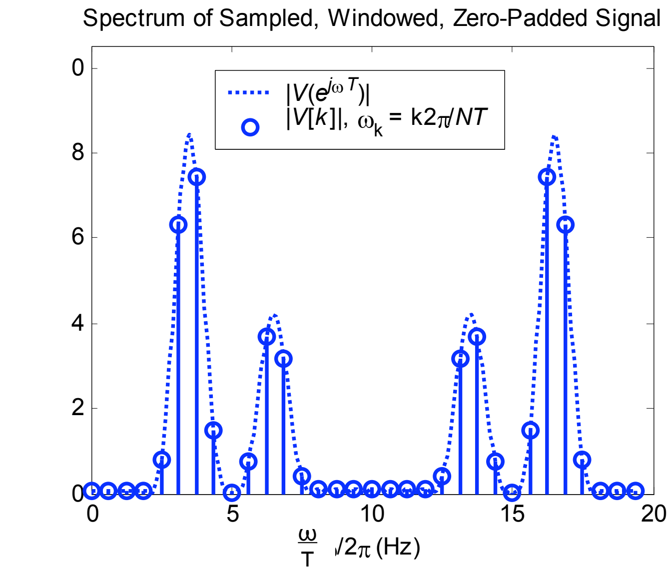
so they are the last $N/2$ samples. (Use `fftshift` to reorder)

Frequency Analysis with DFT Examples:

Hamming Window, $L = 32$, $N = 32$

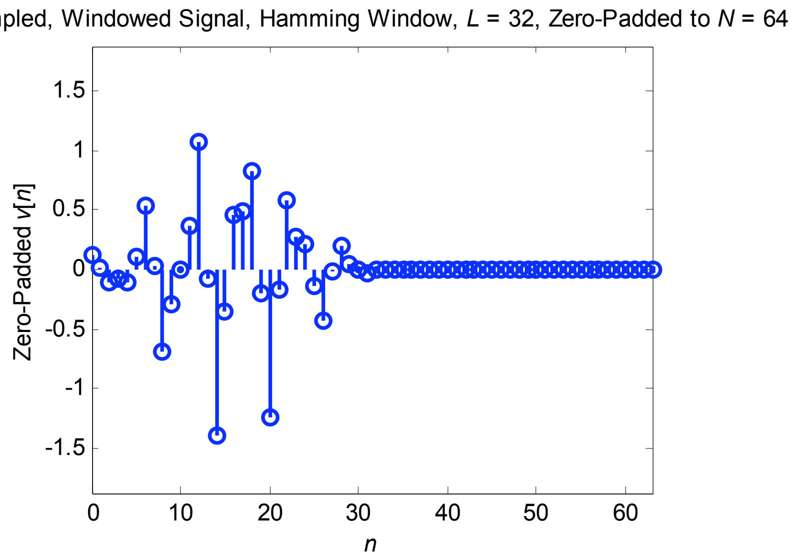


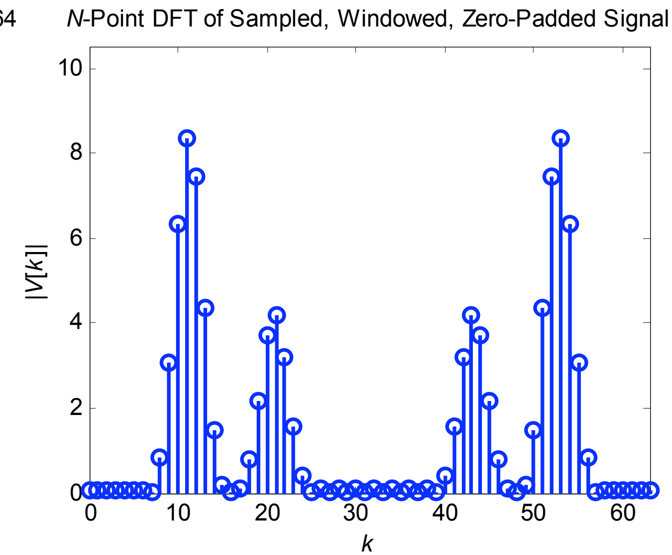


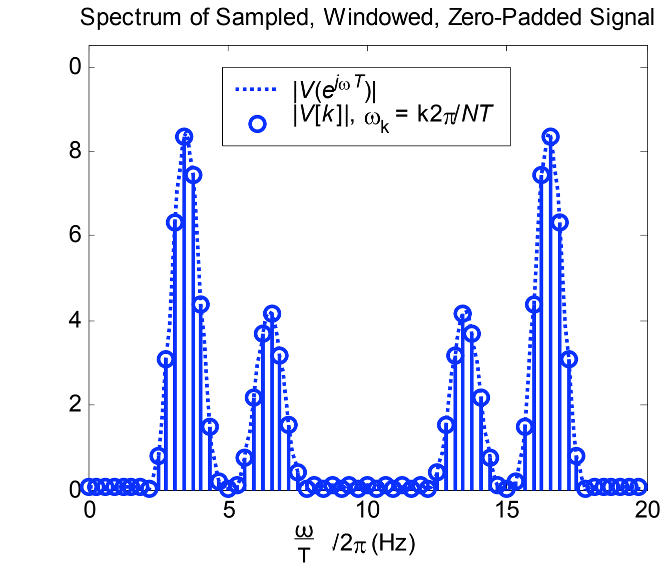


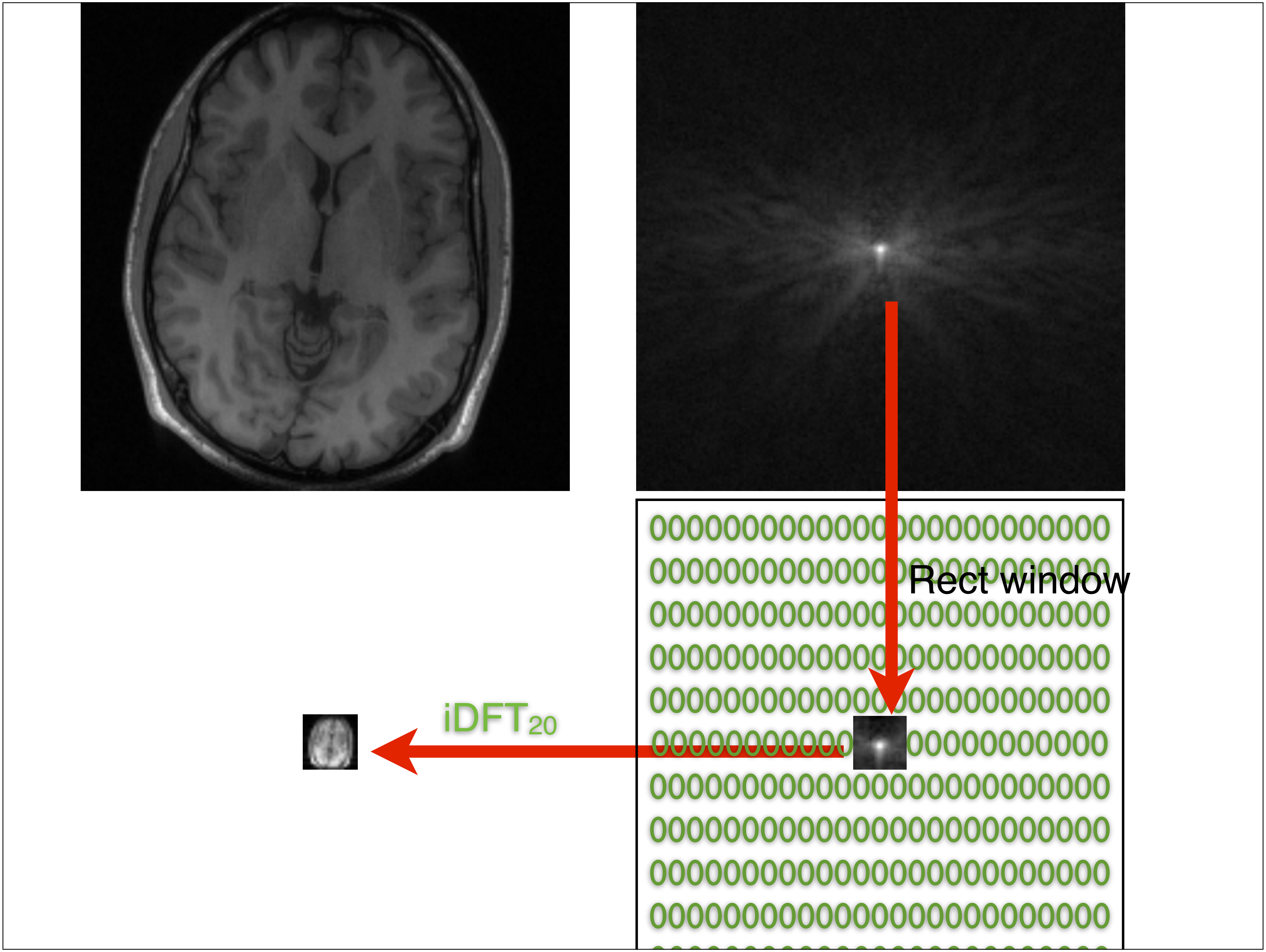
Frequency Analysis with DFT Examples:

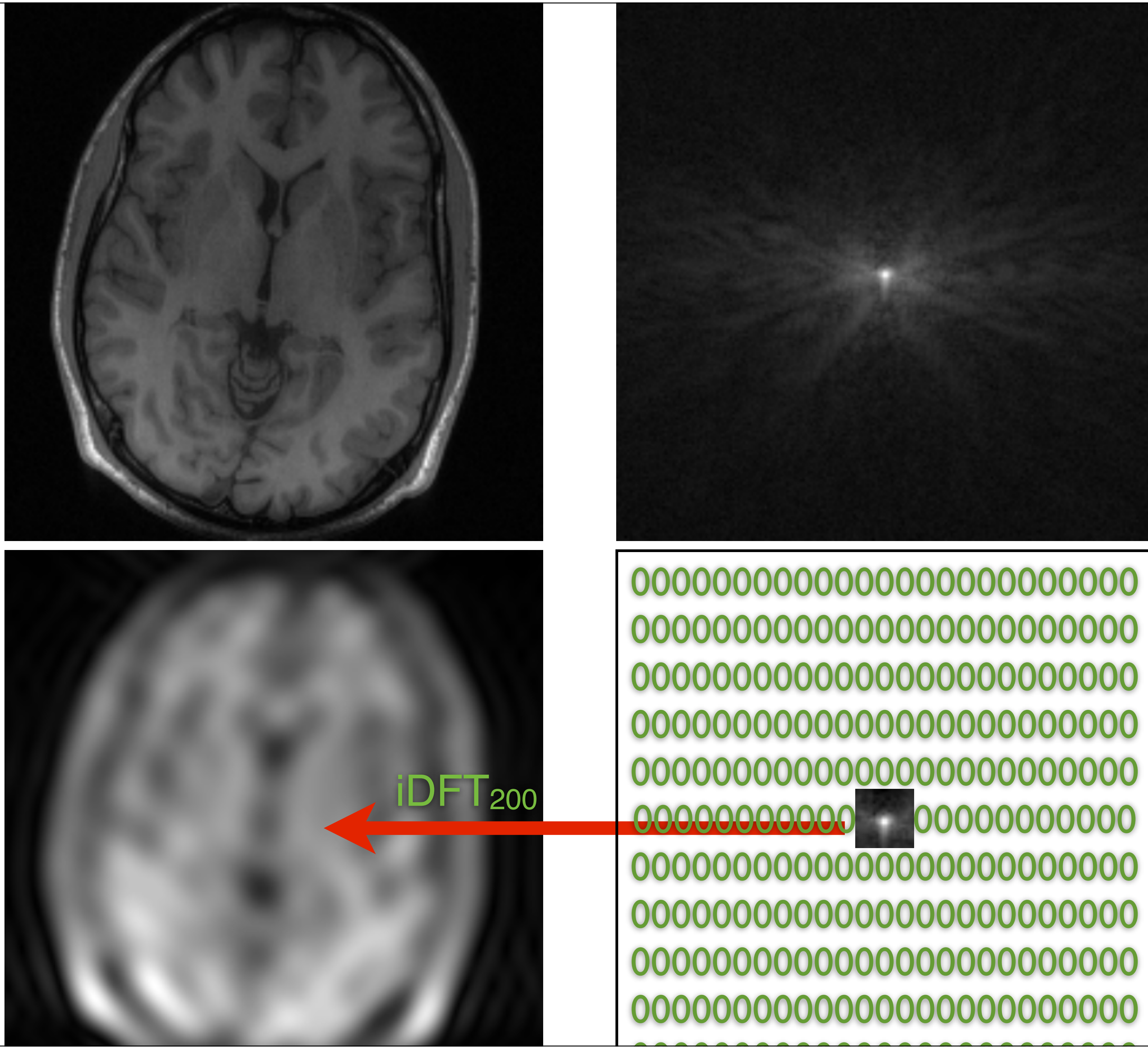
Hamming Window, $L = 32$, Zero-Padded to $N = 64$

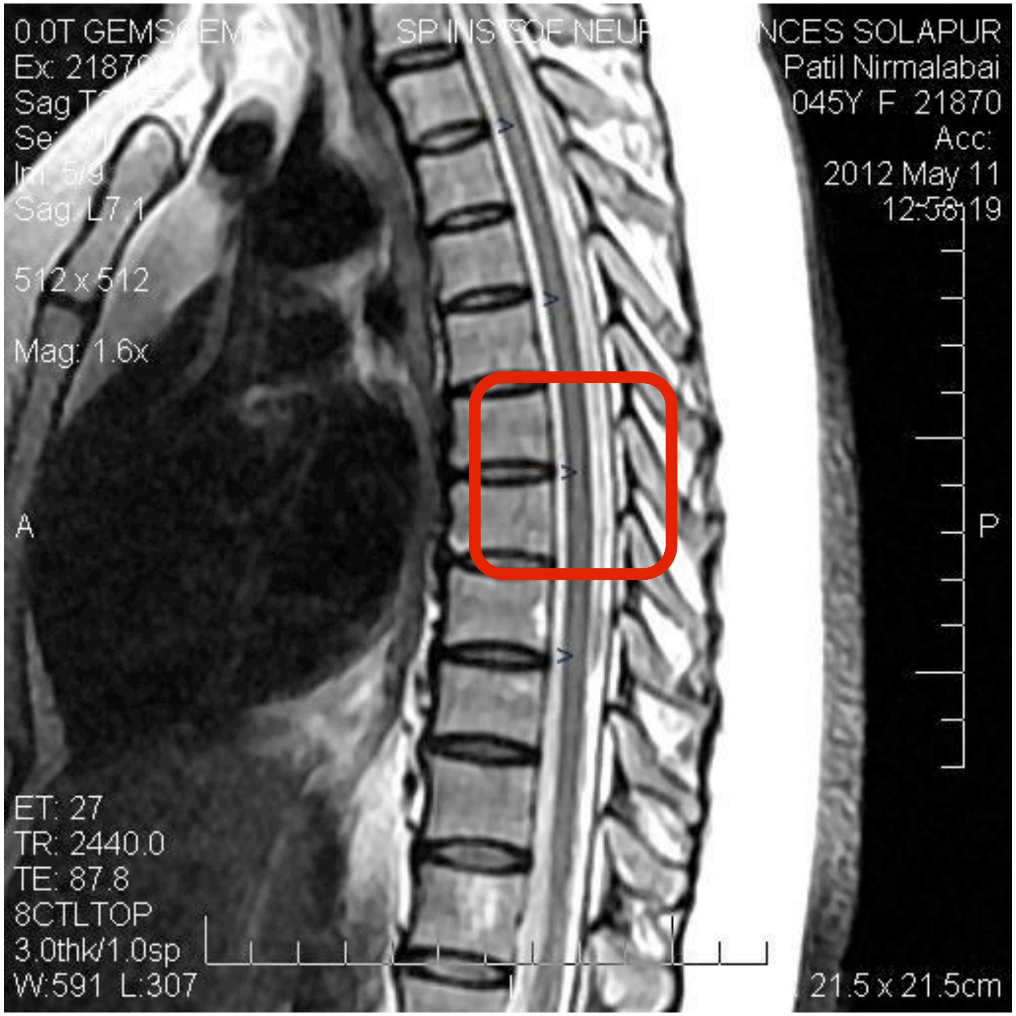


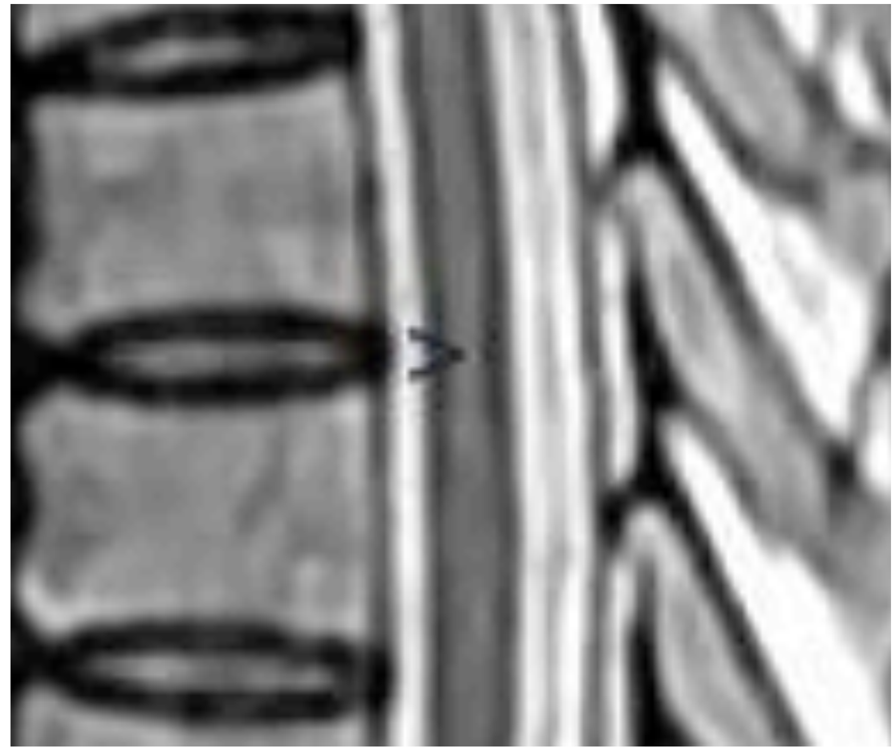





Rect window
iDFT20


iDFT$_{200}$





A 40 yo pt with a history of lower limb weakness referred for mri screening of brain and whole spine for cord. MRI sagittal T2 screening of dorsal region shows a faint uniform linear high signal at the center of the cord. The signal abnormality likely to represent:

(1) Cord demyelination.
(2) Syrinx (spinal cord disease).
(3) Artifact.

Answer : Its an artifact, known as truncation or Gibbs artifact

Frequency Analysis with DFT

- Length of window determines spectral resolution
- Type of window determines side-lobe amplitude. (Some windows have better tradeoff between resolution-sidelobe)
- Zero-padding approximates the DTFT better. Does not introduce new information!

Potential Problems and Solutions

Problem	Possible Solutions
1. Spectral error from aliasing Ch.4	a. Filter signal to reduce frequency content above $\Omega_s/2 = \pi/T$. b. Increase sampling frequency $\Omega_s = 2\pi/T$.
2. Insufficient frequency resolution.	a. Increase L b. Use window having narrow main lobe.
3. Spectral error from leakage	a. Use window having low side lobes. b. Increase L
4. Missing features due to spectral sampling.	a. Increase L, b. Increase N by zero-padding $v[n]$ to length $N > L$.