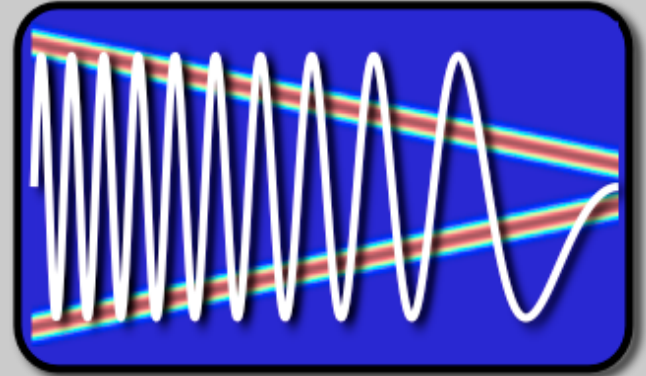


EE123



Digital Signal Processing

Lecture 12

Introduction to Wavelets

Last Time

- Started with STFT
- Heisenberg Boxes

- Continue and move to wavelets

- Ham exam -- see Piazza post
 - Please register at www.eastbayarc.org/form605.htm

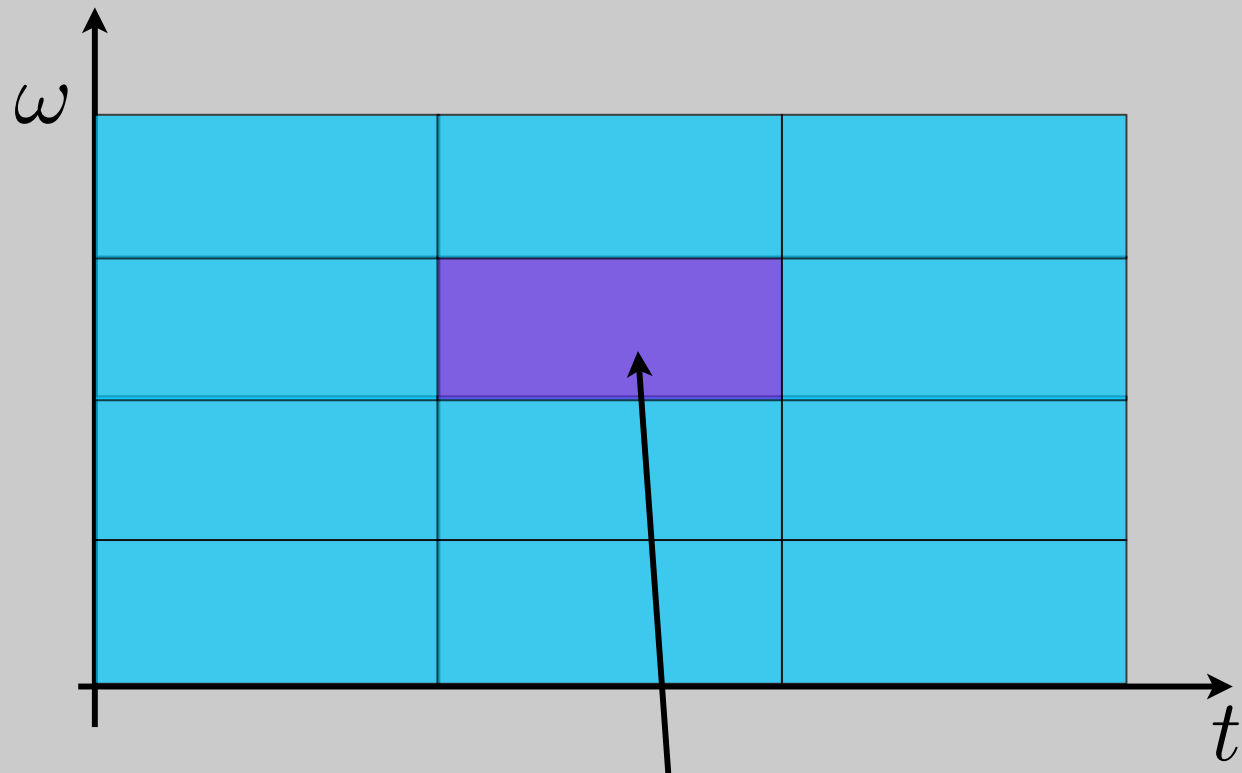
Discrete STFT

$$X[r, k] = \sum_{m=0}^{L-1} x[rR + m] w[m] e^{-j2\pi km/N}$$

optional
↓

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



one STFT coefficient

Limitations of Discrete STFT

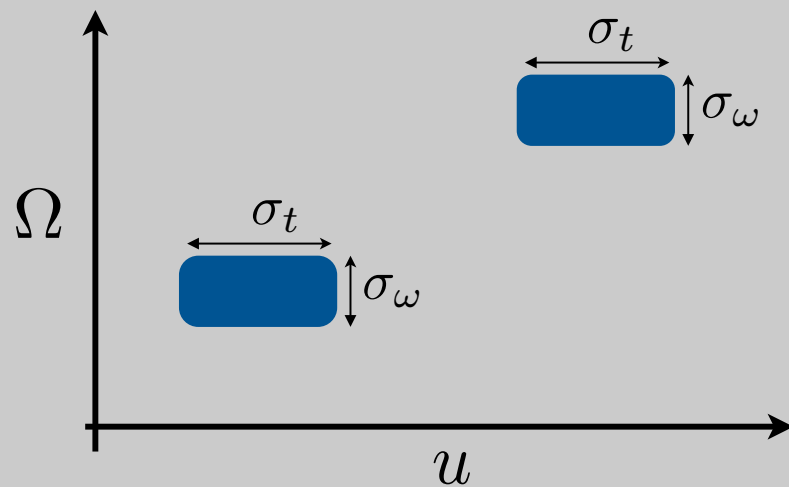
- Need overlapping \Rightarrow Not orthogonal
- Computationally intensive $O(MN \log N)$
- Same size Heisenberg boxes

From STFT to Wavelets

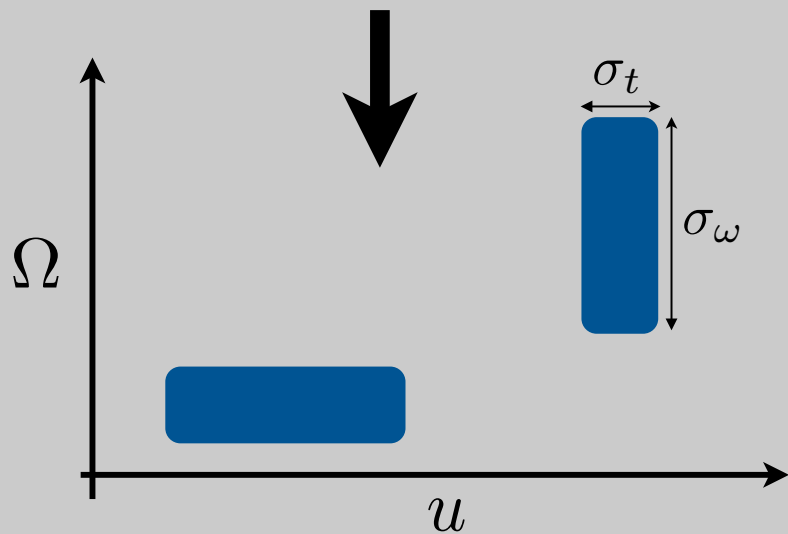
- Basic Idea:
 - low-freq changes slowly - fast tracking unimportant
 - Fast tracking of high-freq is important in many apps.
 - Must adapt Heisenberg box to frequency
- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time



$$Sf(u, \Omega) = \int_{-\infty}^{\infty} f(t)w(t - u)e^{-j\Omega t} dt$$



$$Wf(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*\left(\frac{t - u}{s}\right) dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u, s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^* \left(\frac{t - u}{s} \right) dt$$

- The function Ψ is called a mother wavelet
 - Must satisfy:

$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1 \quad \Rightarrow \text{unit norm}$$

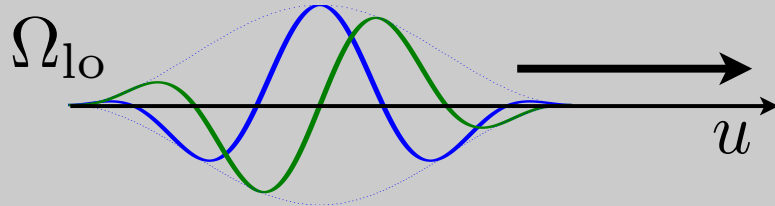
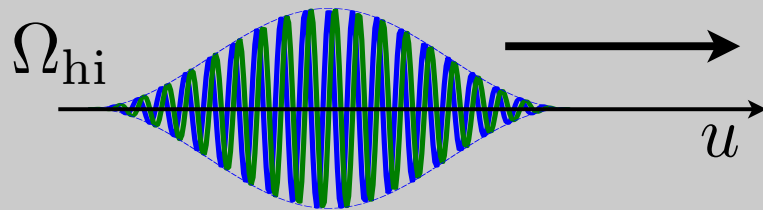
$$\int_{-\infty}^{\infty} \Psi(t) dt = 0 \quad \Rightarrow \text{Band-Pass}$$

STFT and Wavelets “Atoms”

STFT Atoms

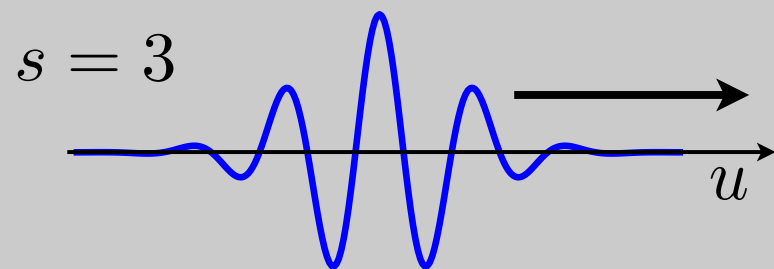
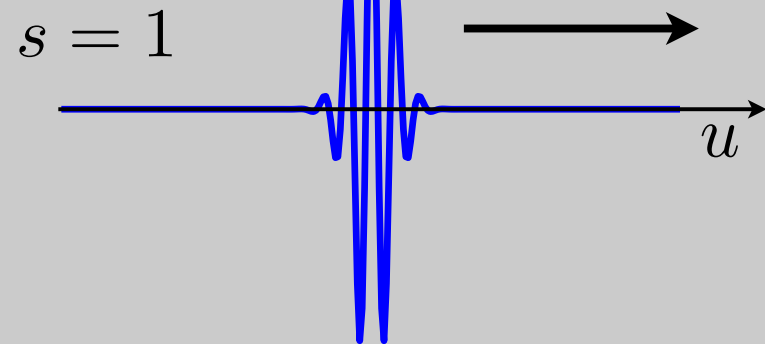
(with hamming window)

$$w(t - u)e^{j\Omega t}$$



Wavelet Atoms

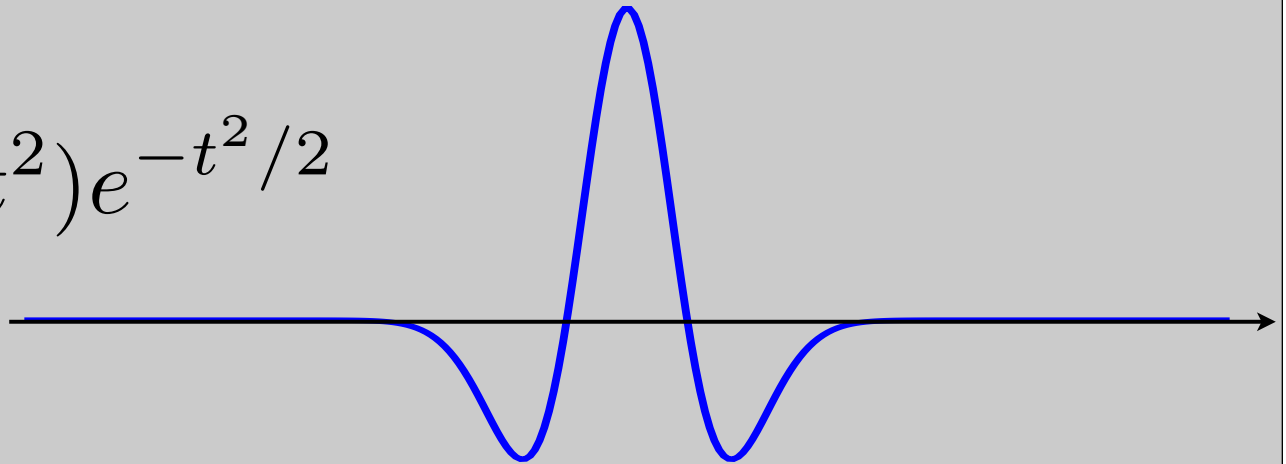
$$\frac{1}{\sqrt{s}} \Psi\left(\frac{t - u}{s}\right)$$



Examples of Wavelets

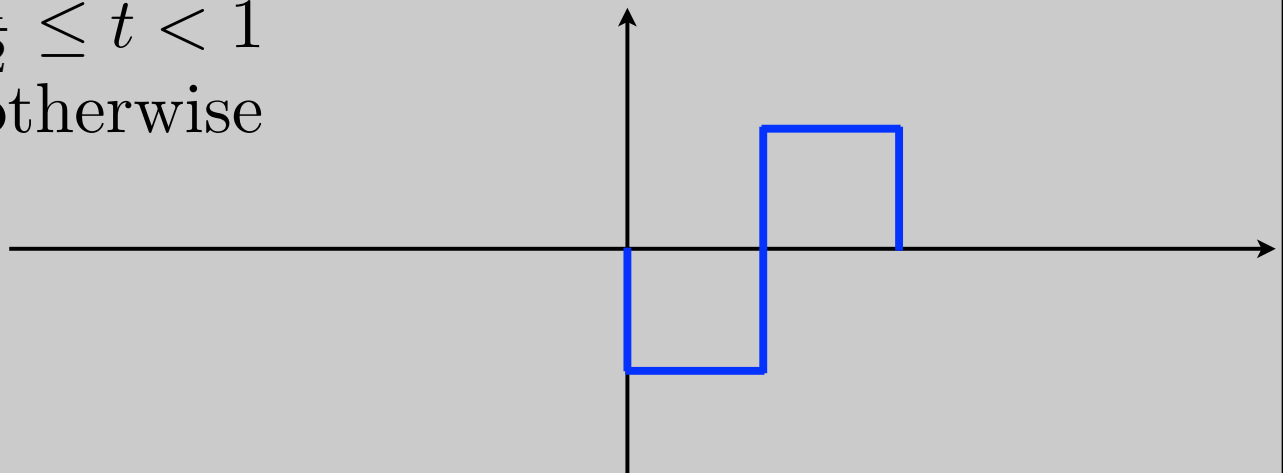
- Mexican Hat

$$\Psi(t) = (1 - t^2)e^{-t^2/2}$$

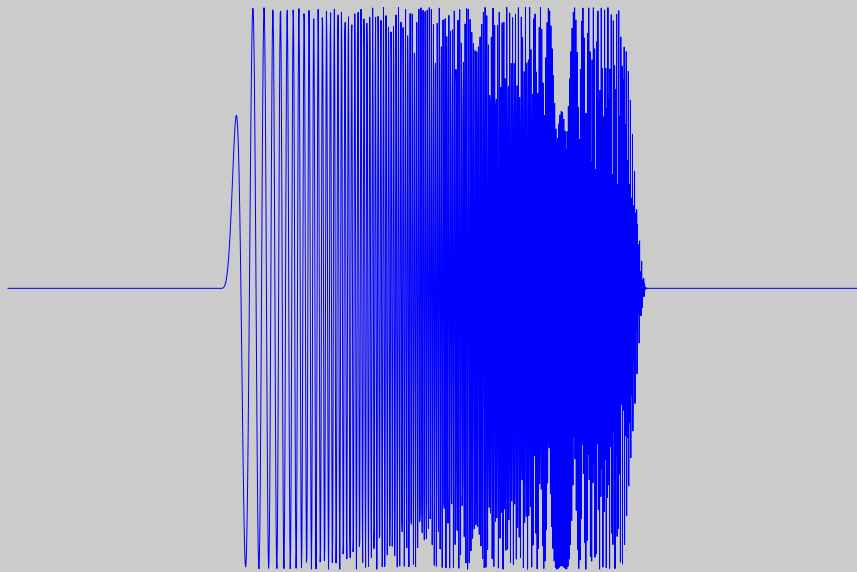


- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

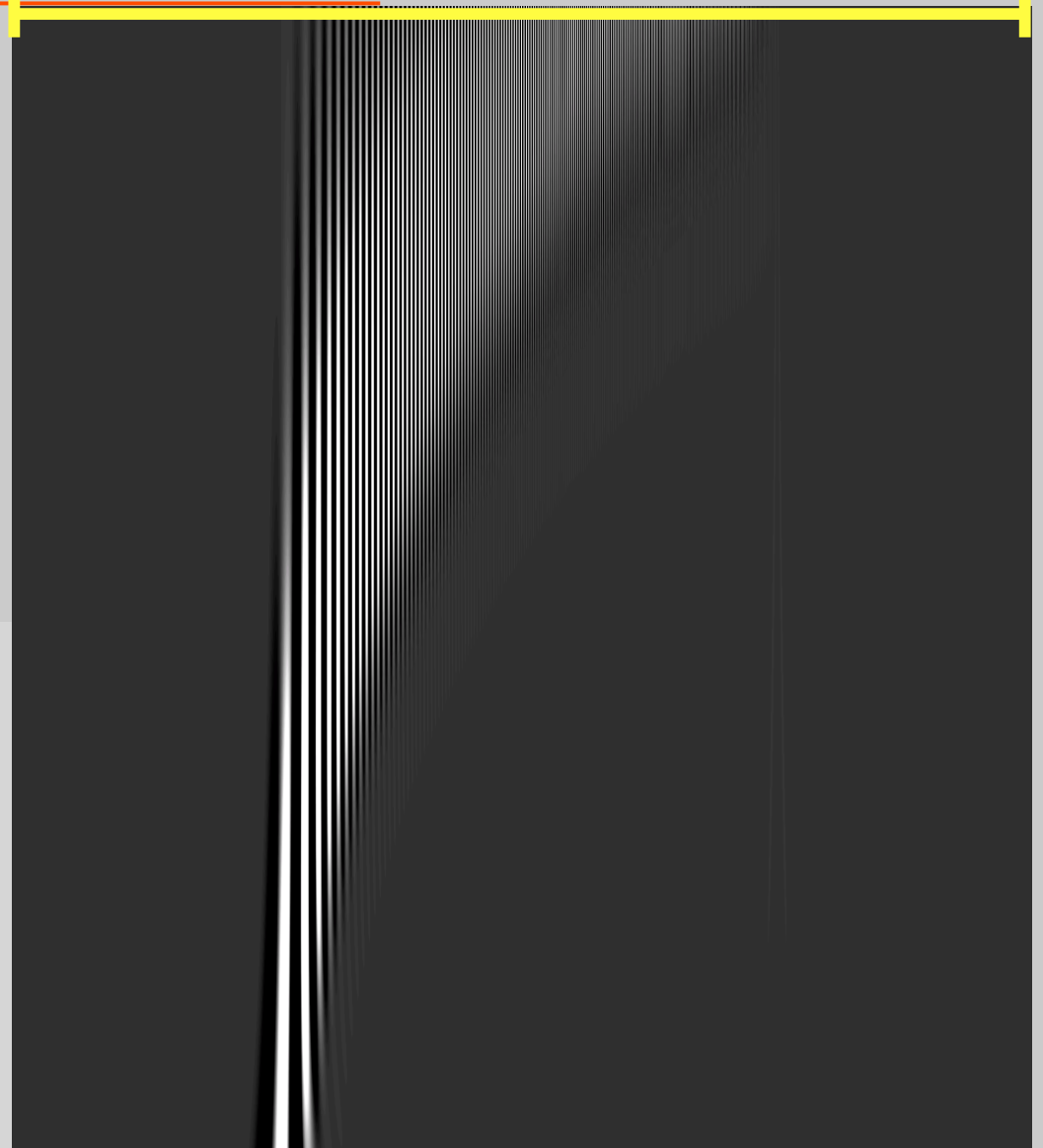
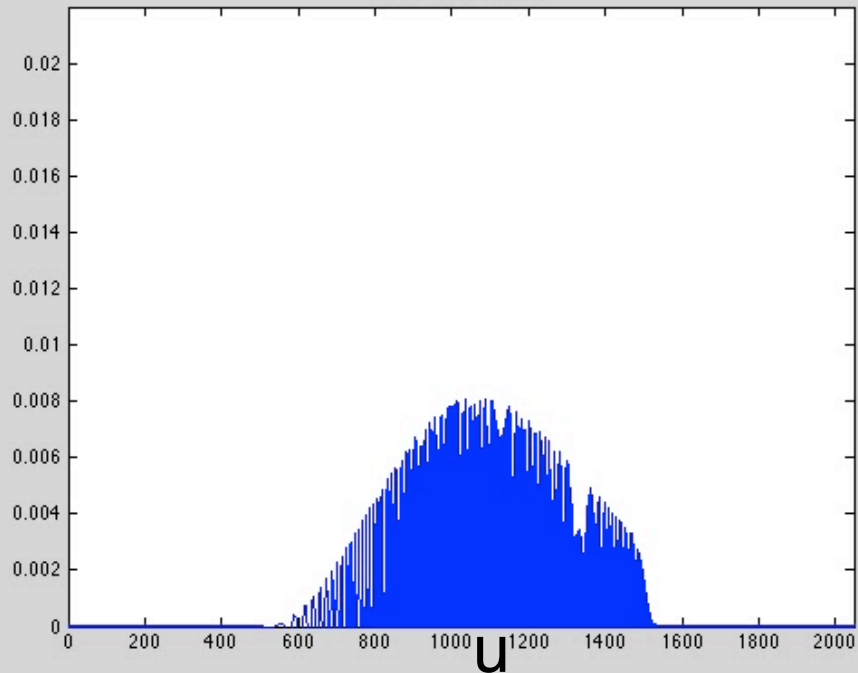


Example: Wavelet of Chirp



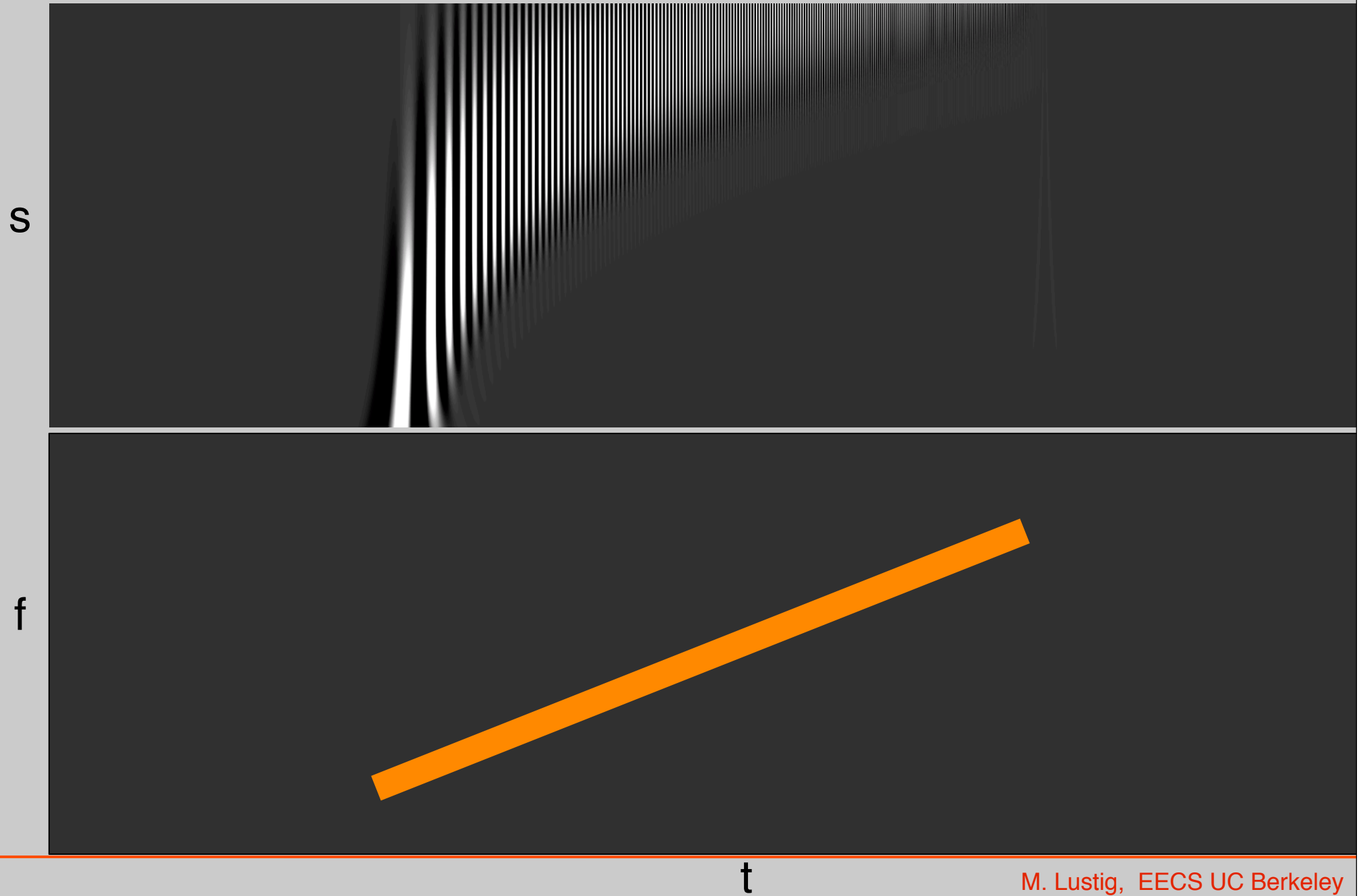
s

Wavelet Coefficients

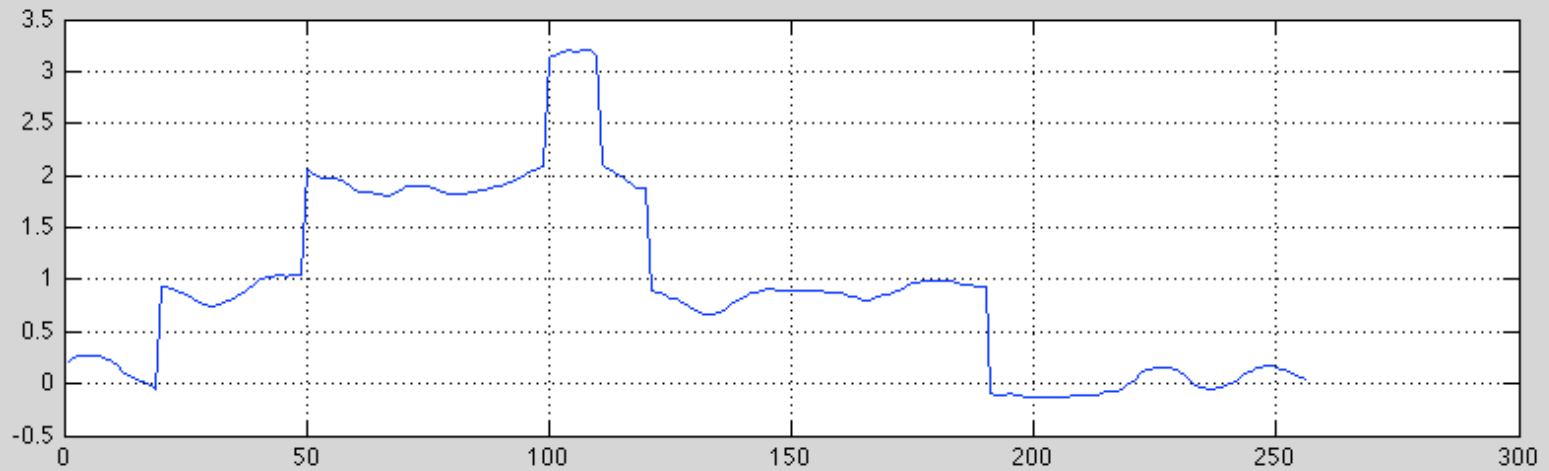


u

Wavelets VS STFT



Example 2: "Bumpy" Signal



SombreroWavelet

$\log(s)$

u

Wavelets Transform

- Can be written as linear filtering

$$\begin{aligned} Wf(u, s) &= \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t) \Psi^* \left(\frac{t - u}{s} \right) dt \\ &= \{ f(t) * \bar{\Psi}_s(t) \} (u) \end{aligned}$$

$$\bar{\Psi}_s = \frac{1}{\sqrt{s}} \Psi \left(\frac{t}{s} \right)$$

- Wavelet coefficients are a result of bandpass filtering

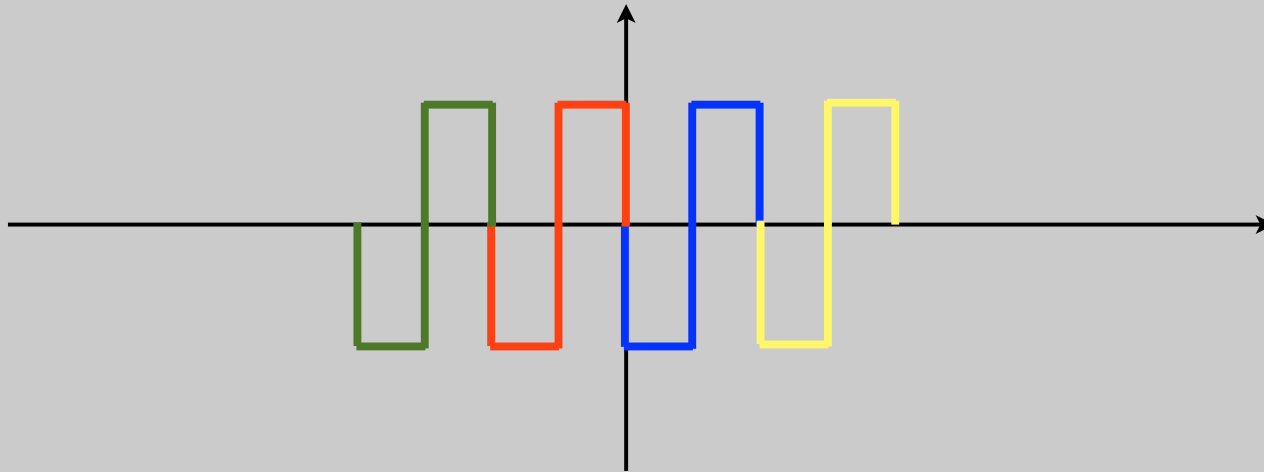
Wavelet Transform

- Many different constructions for different signals
 - Haar good for piece-wise constant signals
 - Battle-Lemarie' : Spline polynomials
- Can construct Orthogonal wavelets
 - For example: dyadic Haar is orthonormal

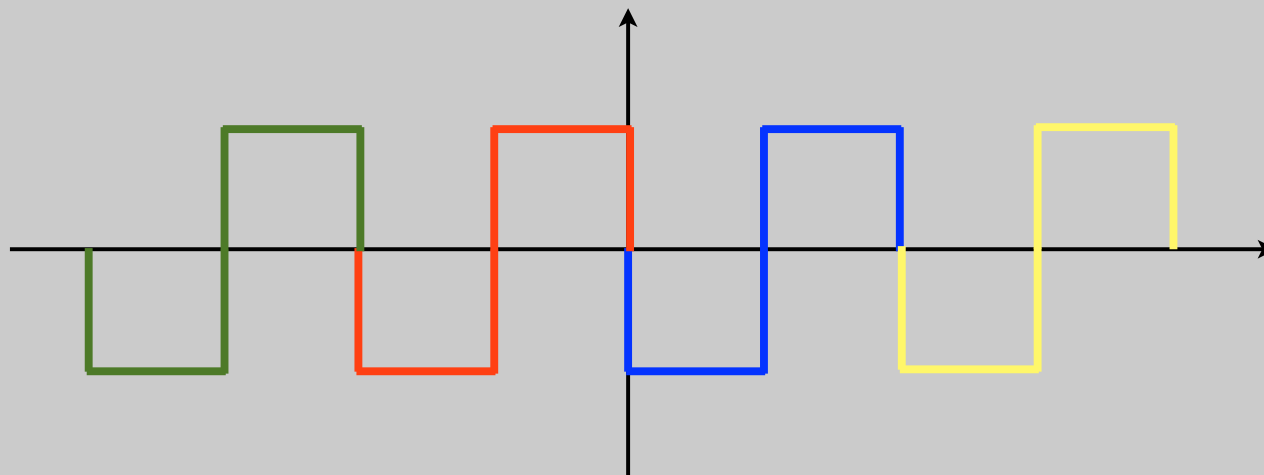
$$\bar{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$

$i = [1, 2, 3, \dots]$

Orthonormal Haar



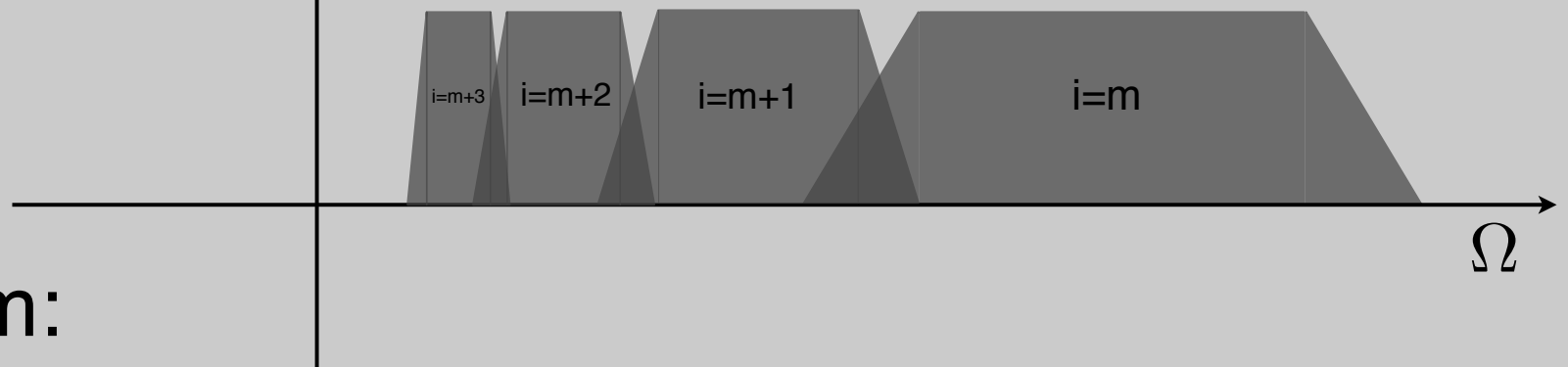
Same scale
non-overlapping



Orthogonal
between scales

Scaling function

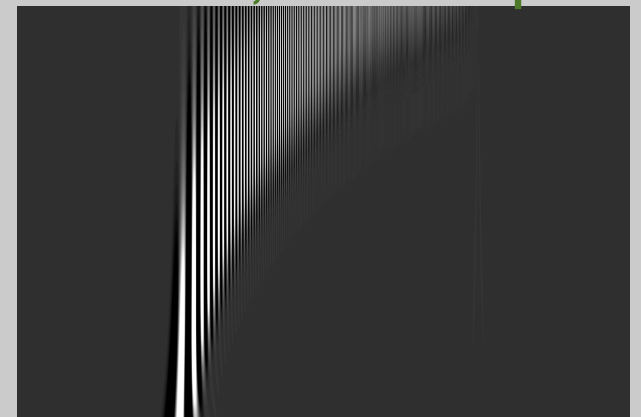
$$\bar{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$



- Problem:

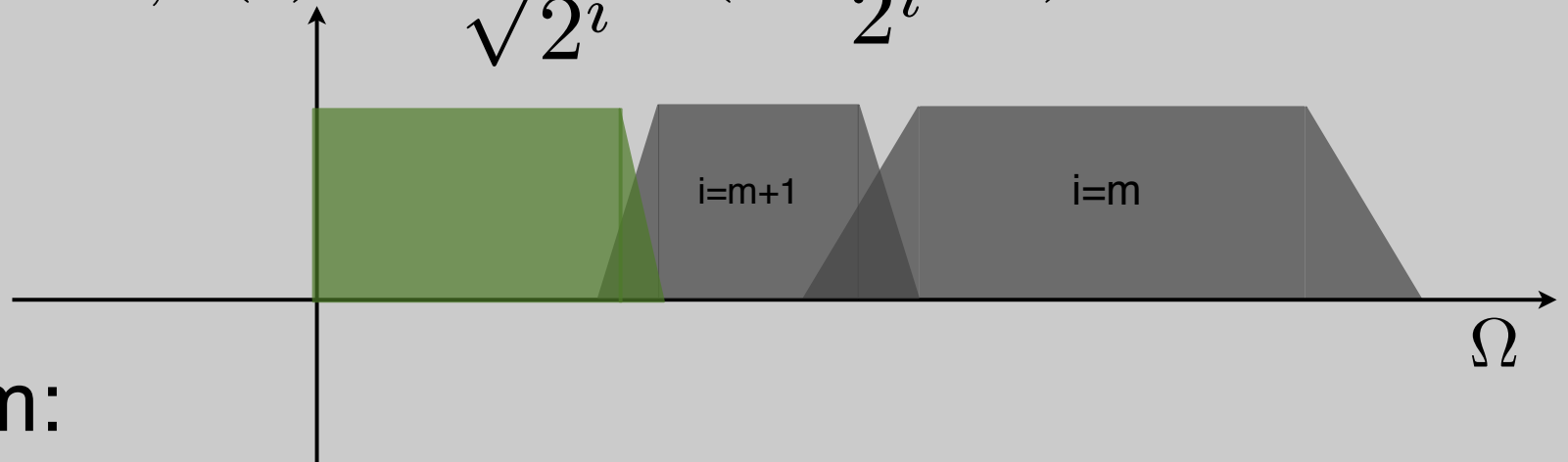
- Every stretch only covers half remaining bandwidth
- Need Infinite functions

recall, for chirp:



Scaling function

$$\bar{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}} \Psi\left(\frac{t - 2^i n}{2^i}\right)$$



- **Problem:**

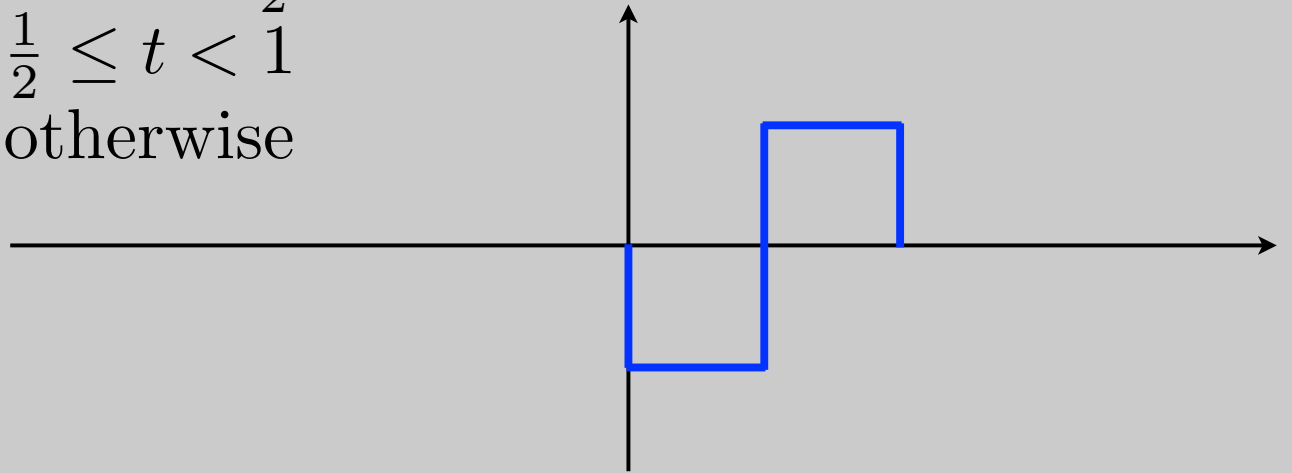
- Every stretch only covers half remaining bandwidth
- Need Infinite functions

- **Solution:**

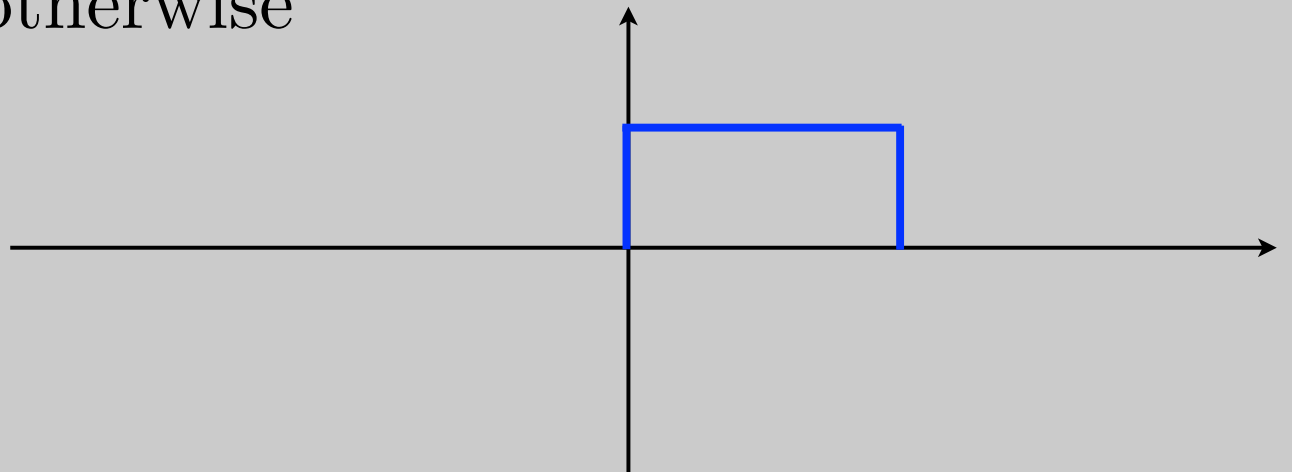
- Plug low-pass spectrum with a scaling function $\bar{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Back to Discrete

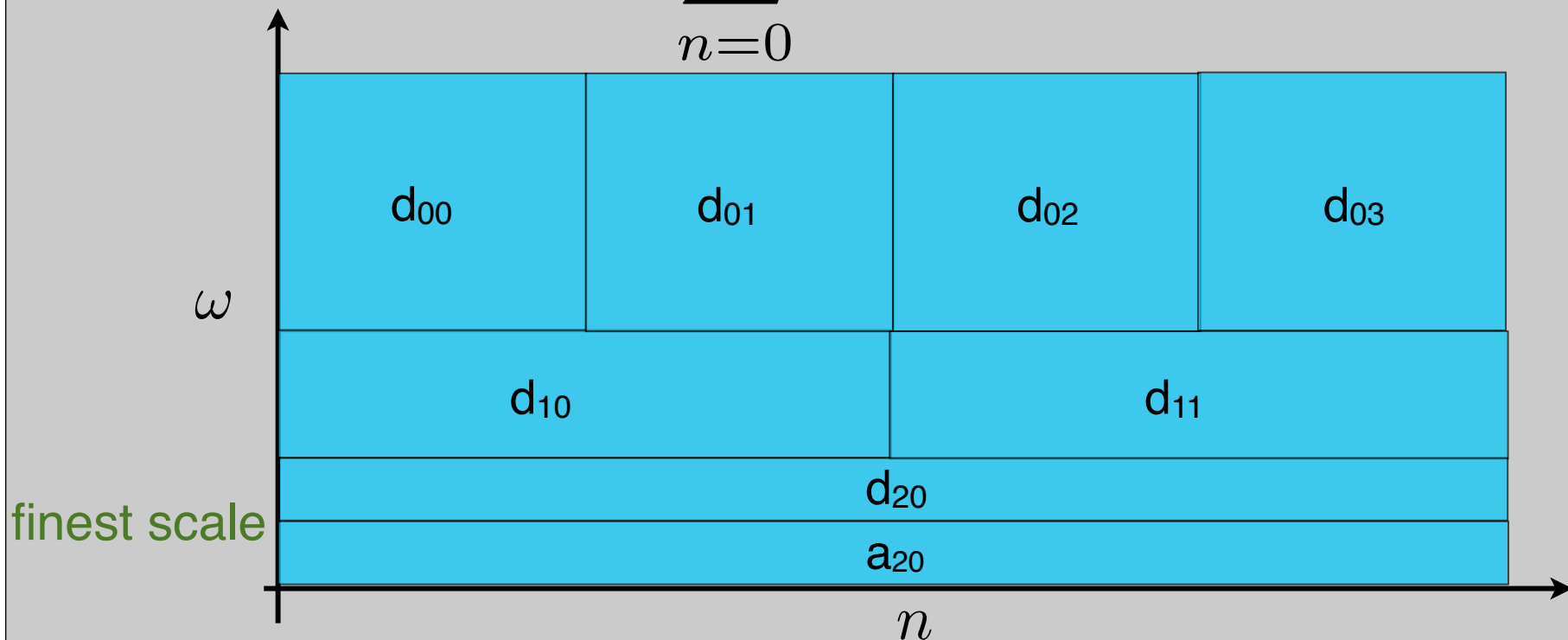
- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties

Discrete Wavelet Transform

$$d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n]$$

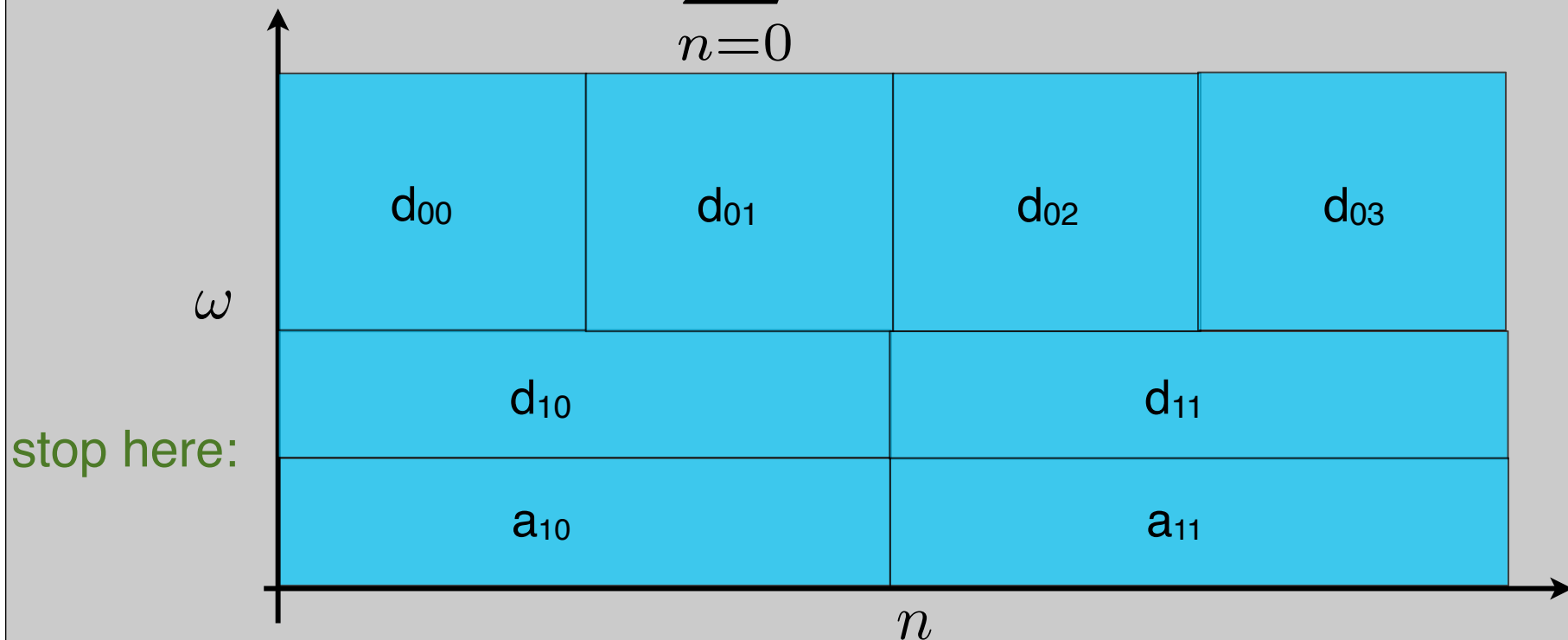
$$a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n]$$



Discrete Wavelet Transform

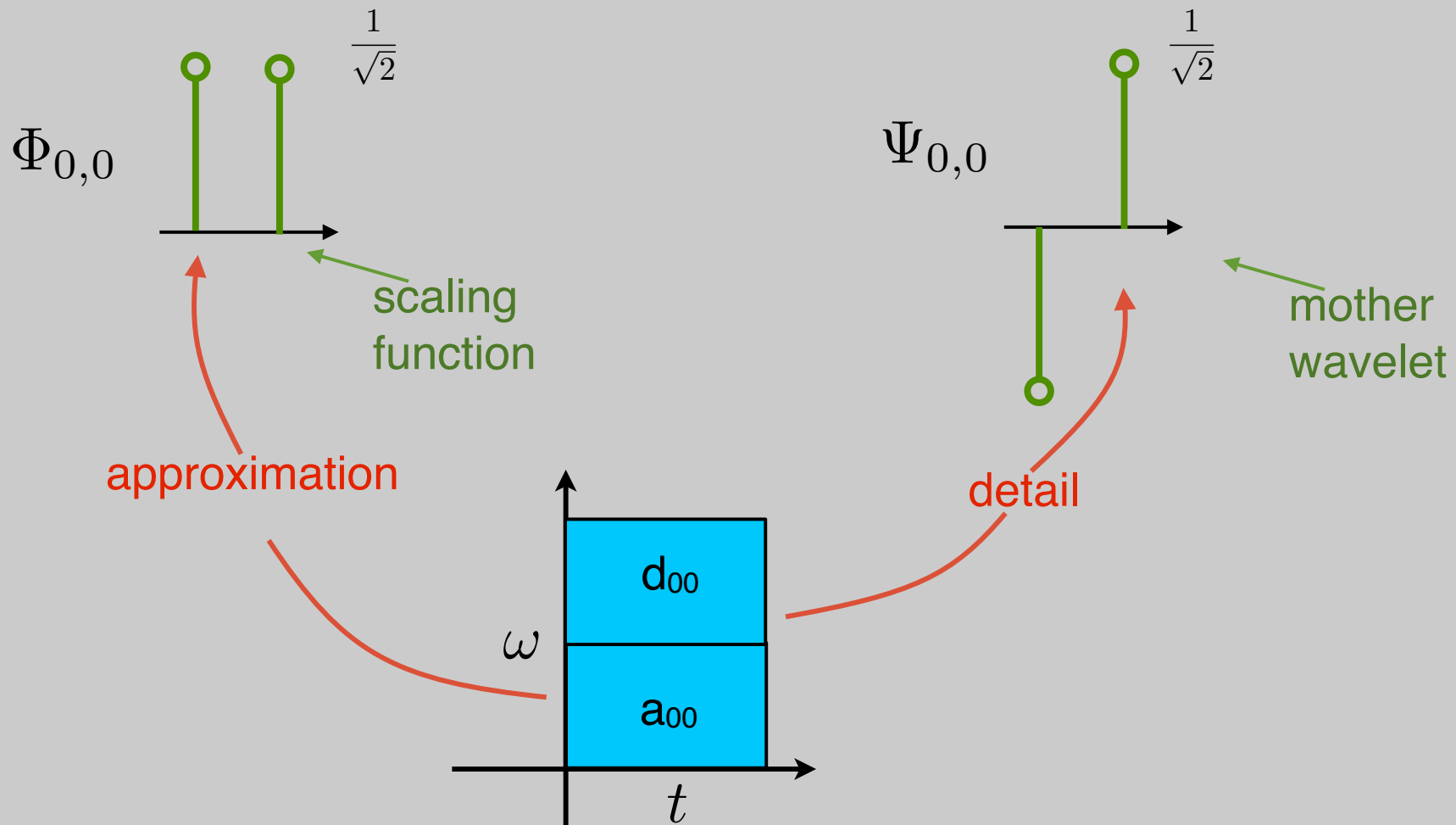
$$d_{s,u} = \sum_{n=0}^{N-1} x[n] \Psi_{s,u}[n]$$

$$a_{s,u} = \sum_{n=0}^{N-1} x[n] \Phi_{s,u}[n]$$



Example: Discrete Haar Wavelet

Haar for $n=2$



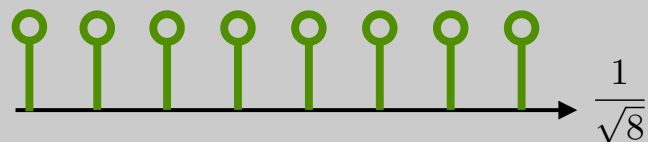
Equivalent to DFT_2 !

Discrete Orthogonal Haar Wavelet

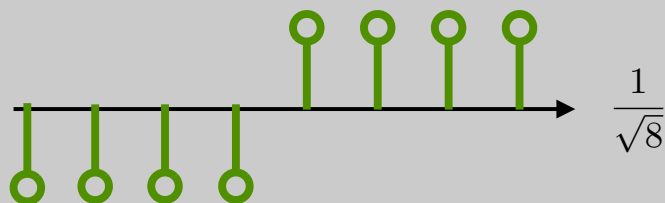
Haar for n=8

scaling

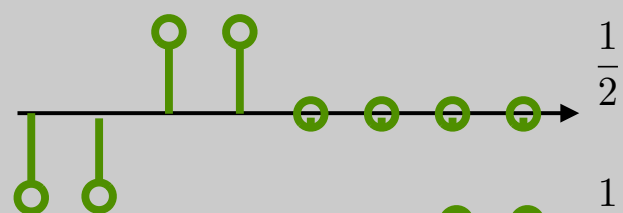
Φ_{20}



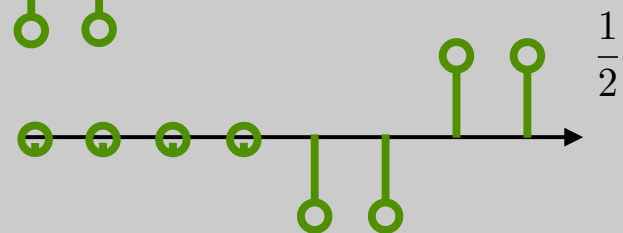
Ψ_{20}



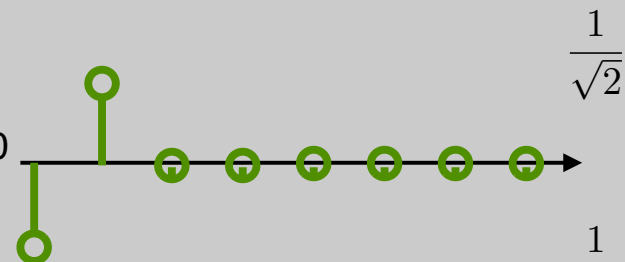
Ψ_{10}



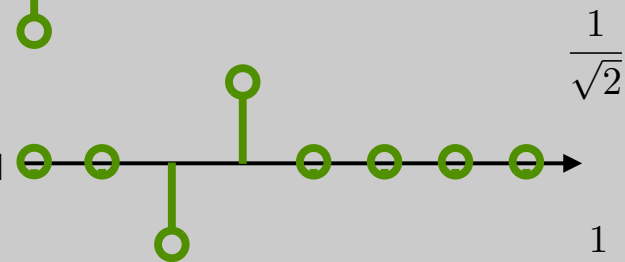
Ψ_{11}



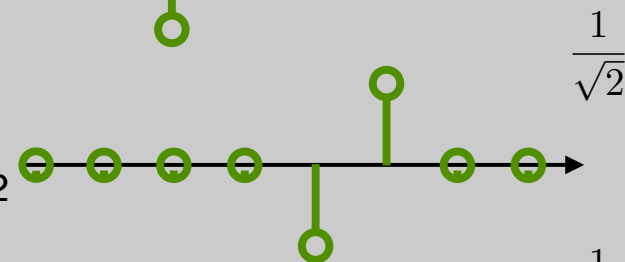
Ψ_{00}



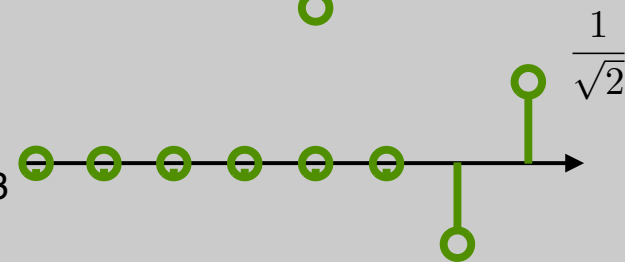
Ψ_{01}



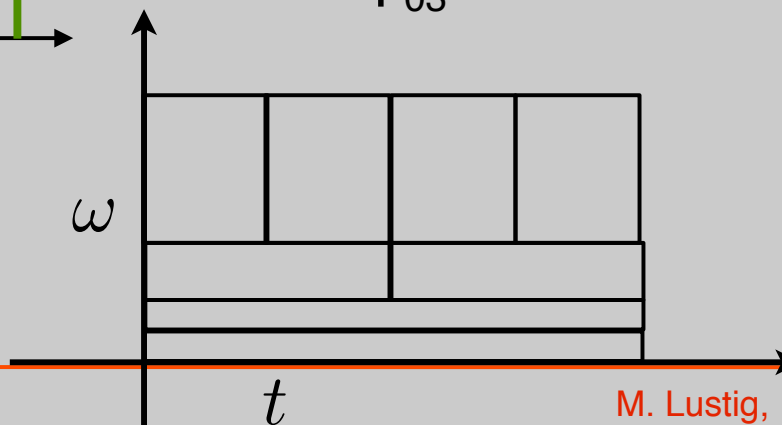
Ψ_{02}



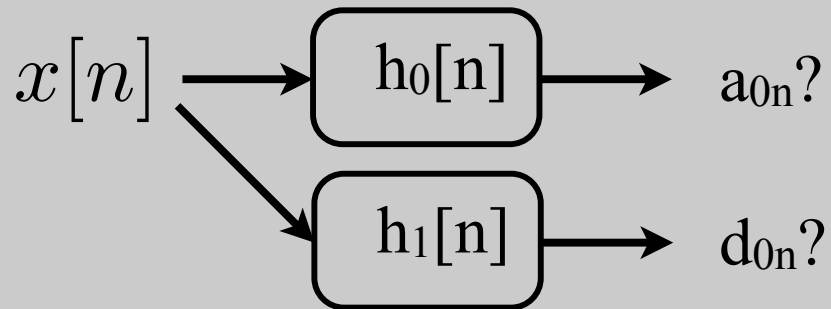
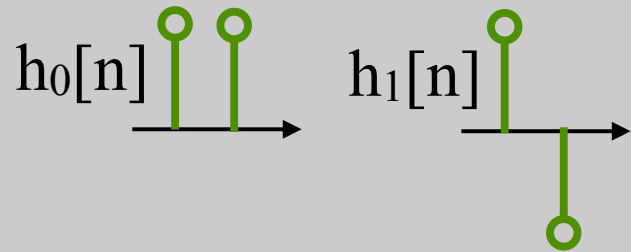
Ψ_{03}



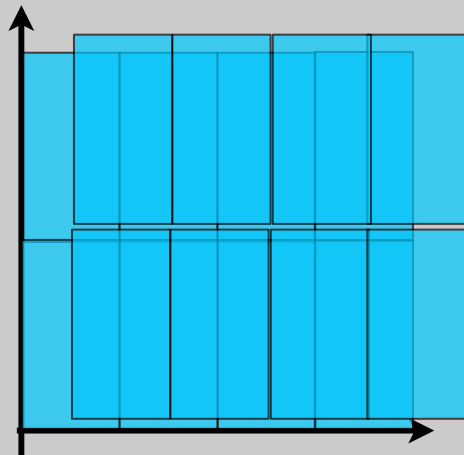
ω



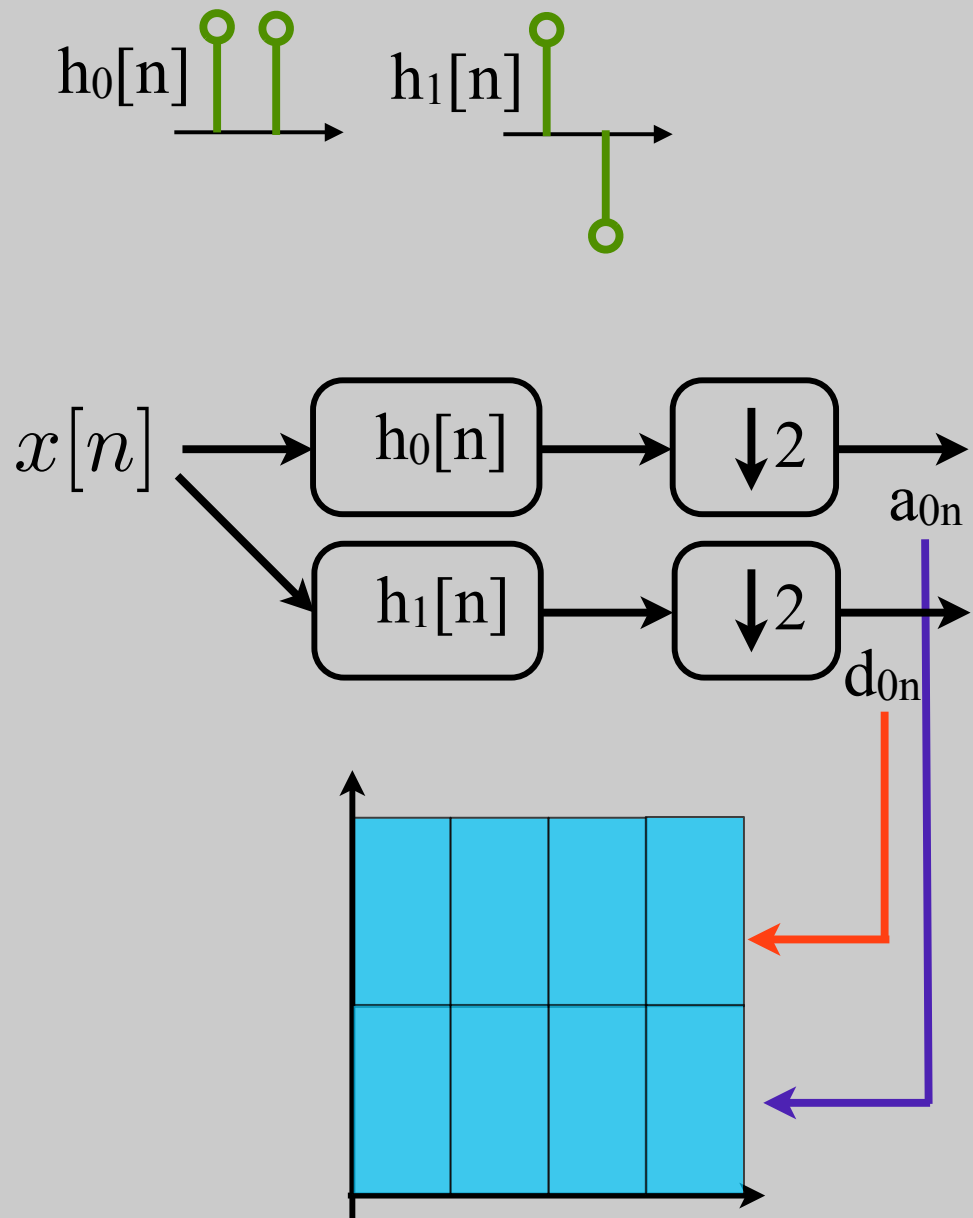
Fast DWT with Filter Banks (more Later!)



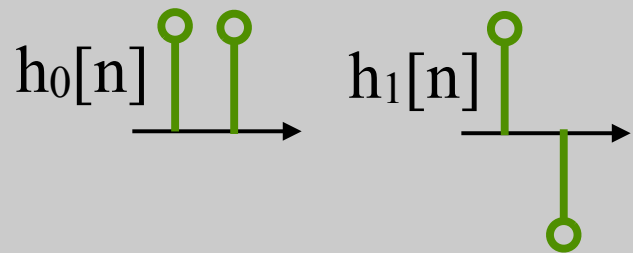
not quite...
too many
coefficients



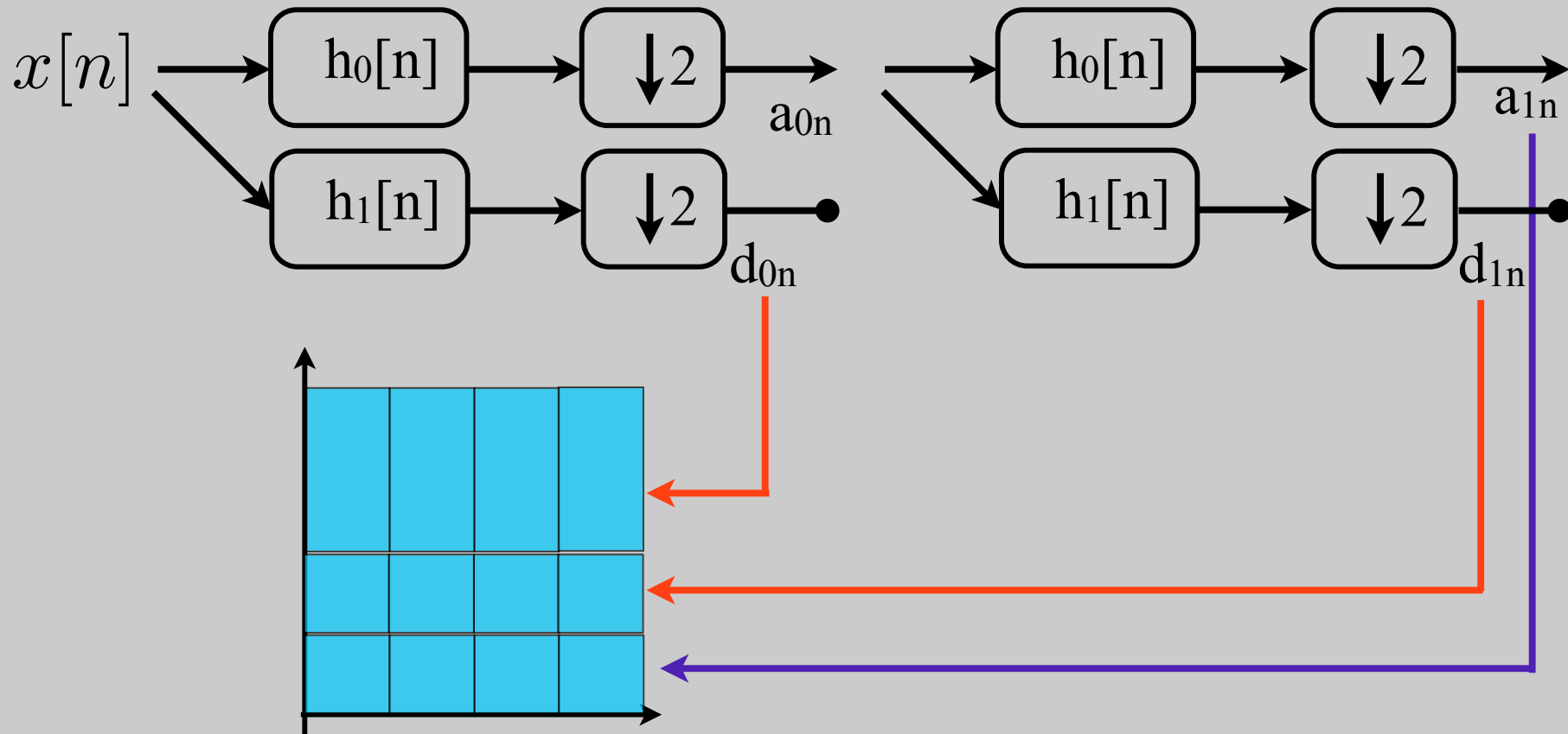
Fast DWT with Filter Banks



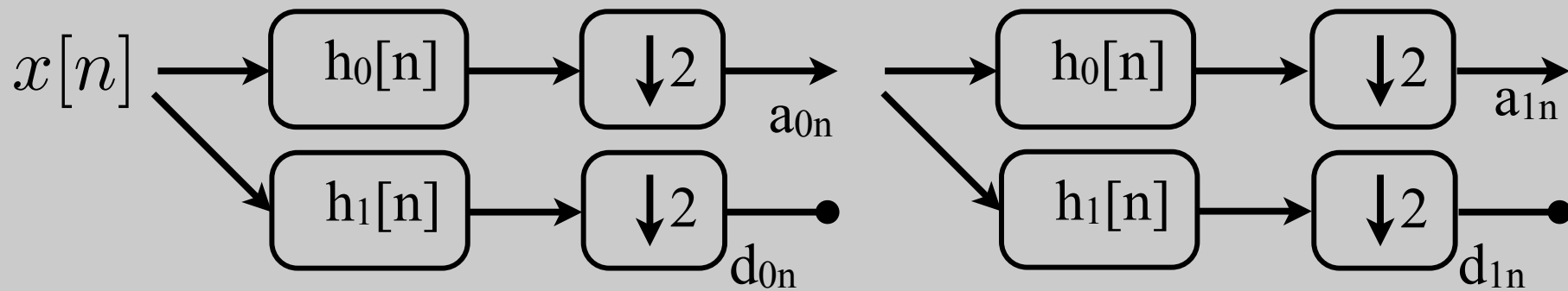
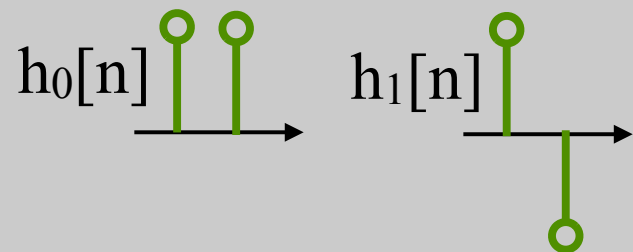
Fast DWT with Filter Banks



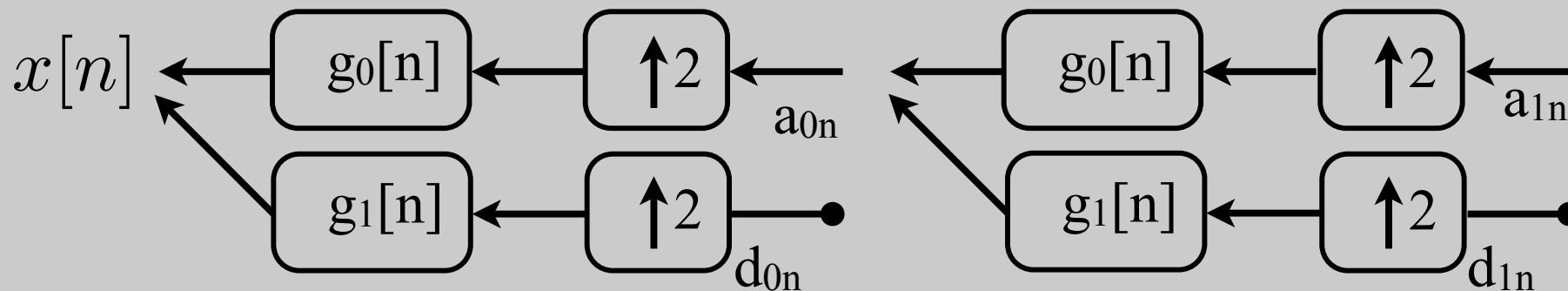
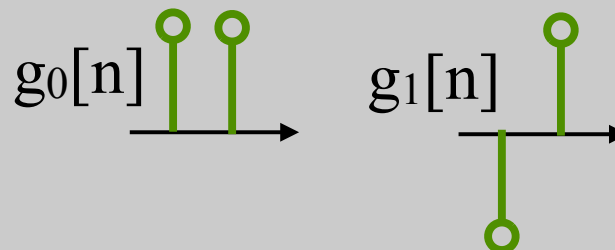
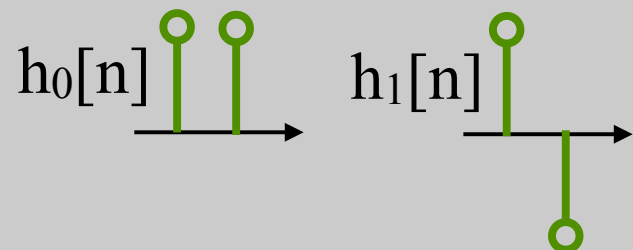
complexity:
 $N + N/2 + N/4 + N/8 + \dots + = 2N$
 $= O(N)$



Decomposition

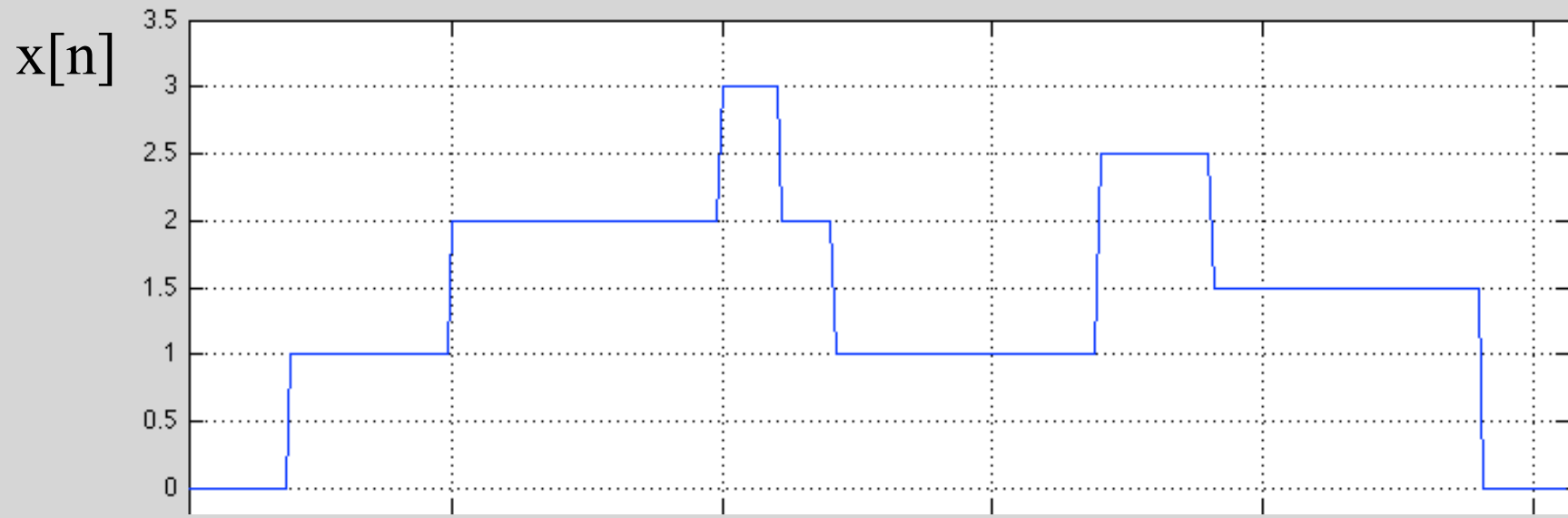


Reconstruction

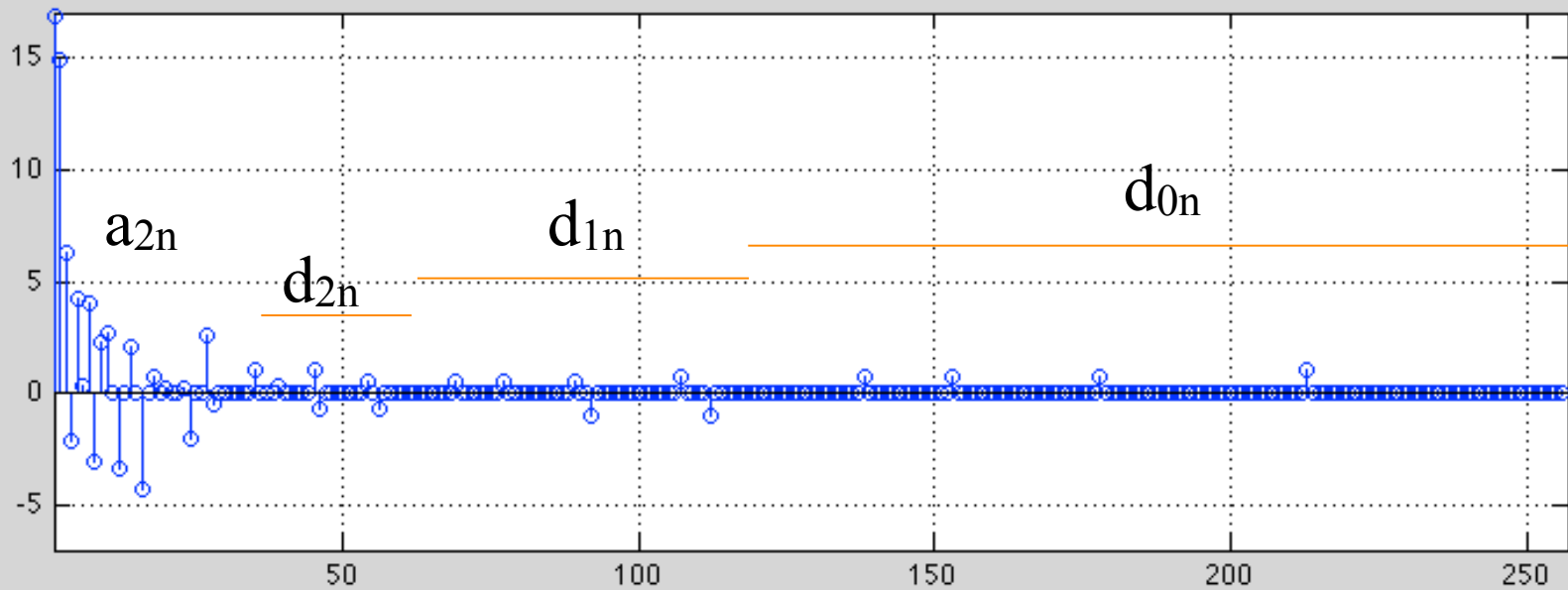


Just flip arrows,
replace h with g

Haar DWT Example

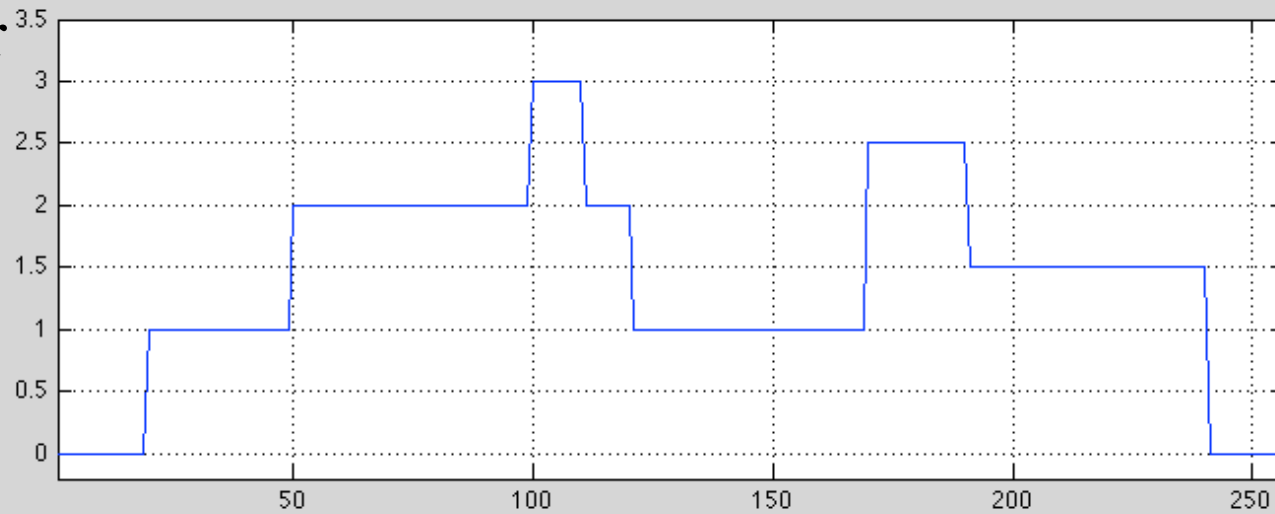


Haar

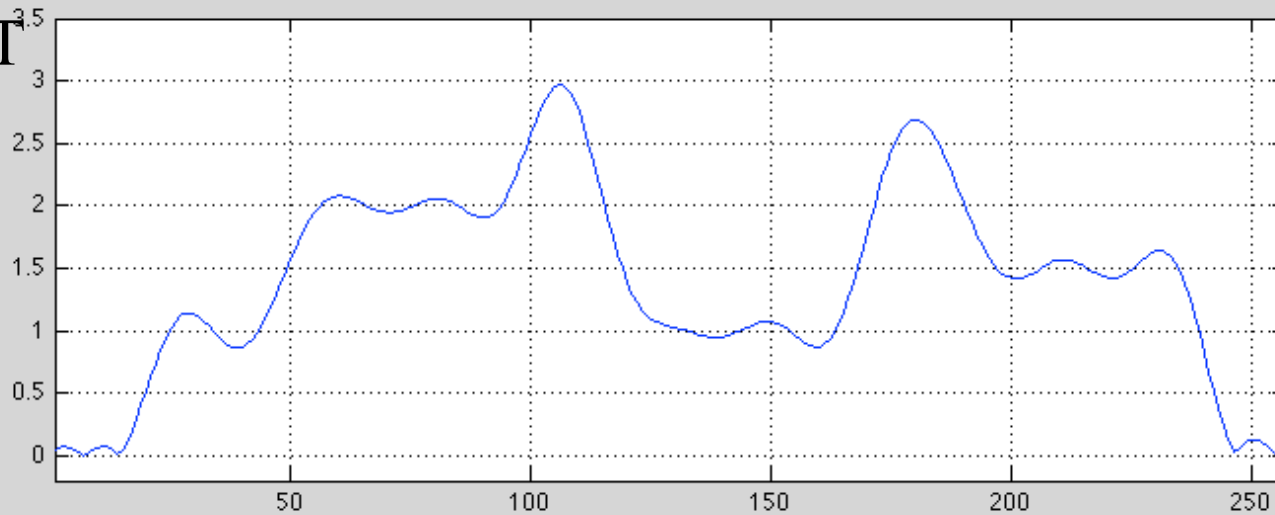


Approximation from 25/256 coefficients

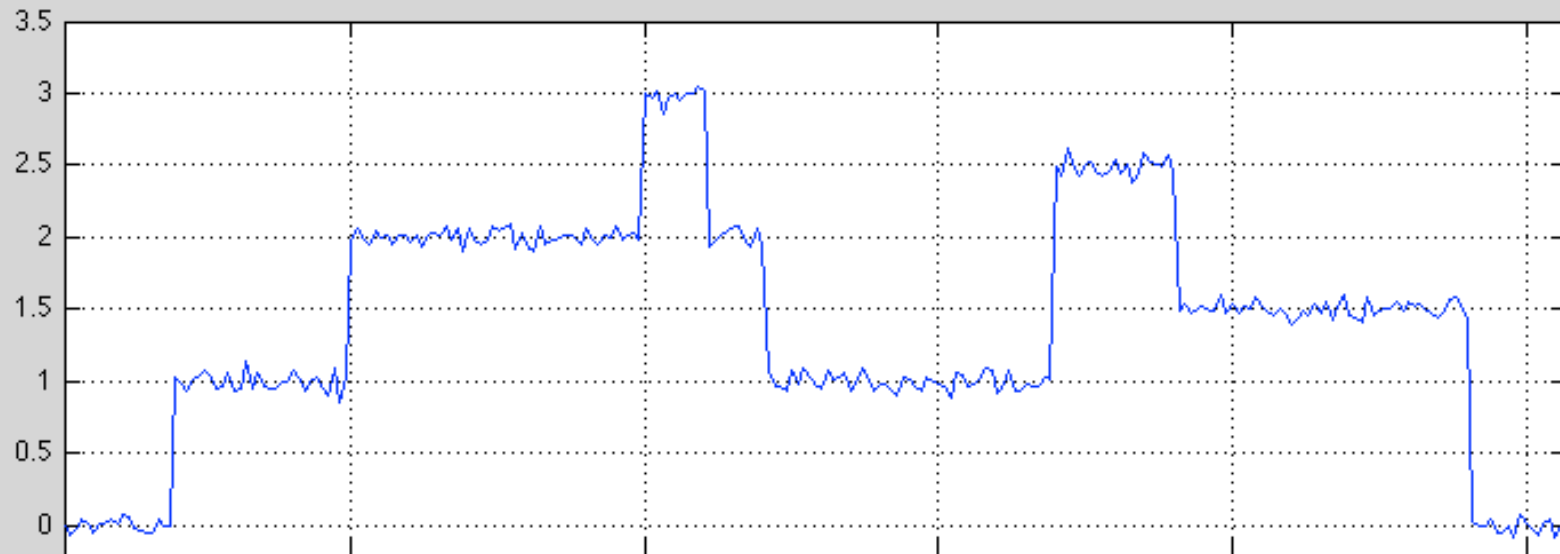
Haar



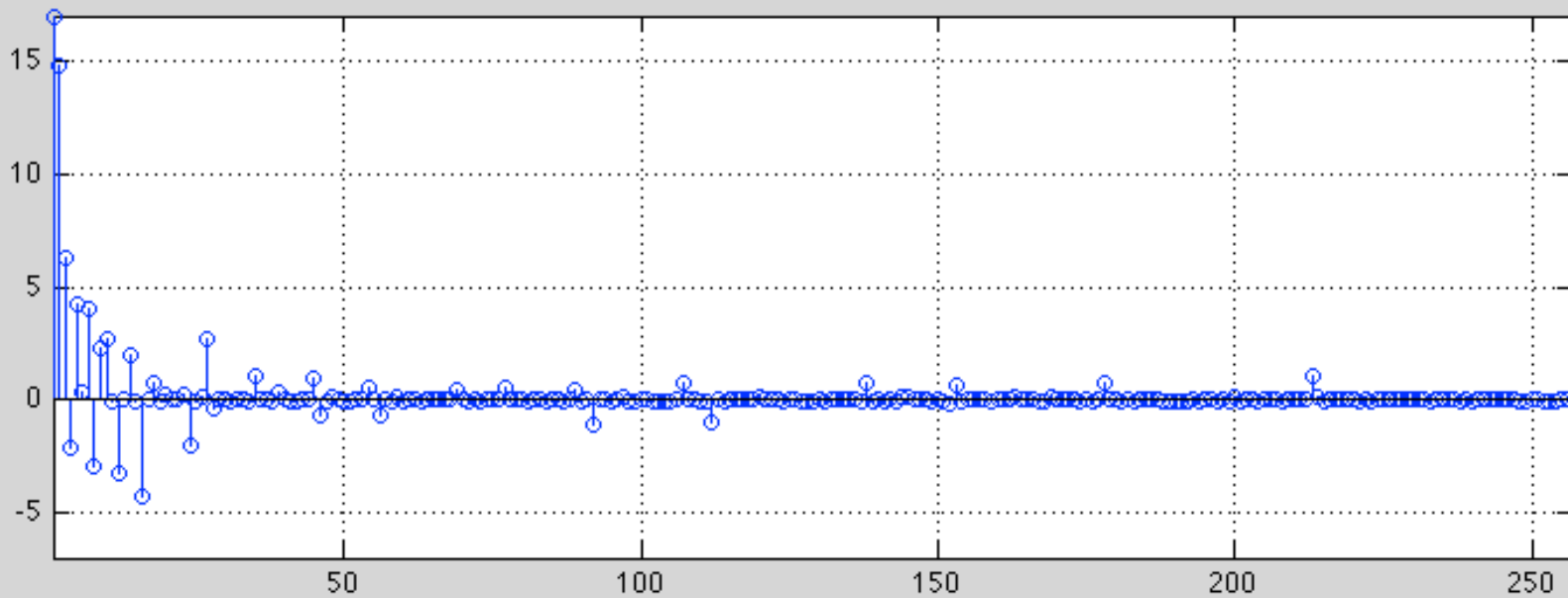
DFT



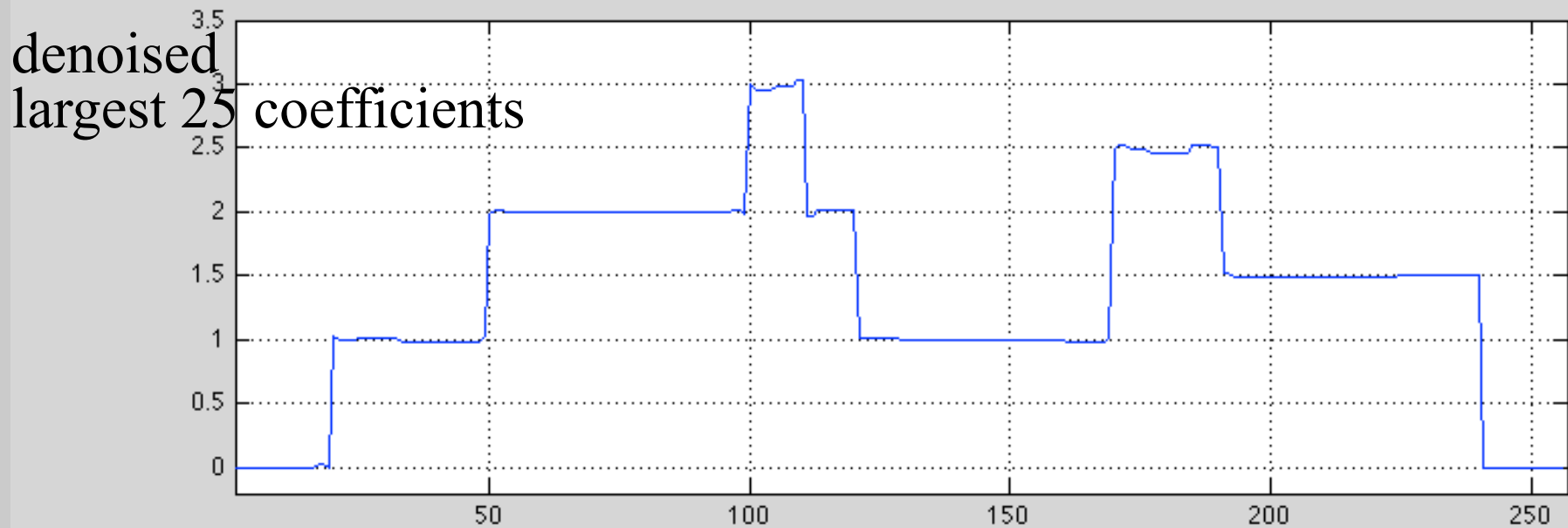
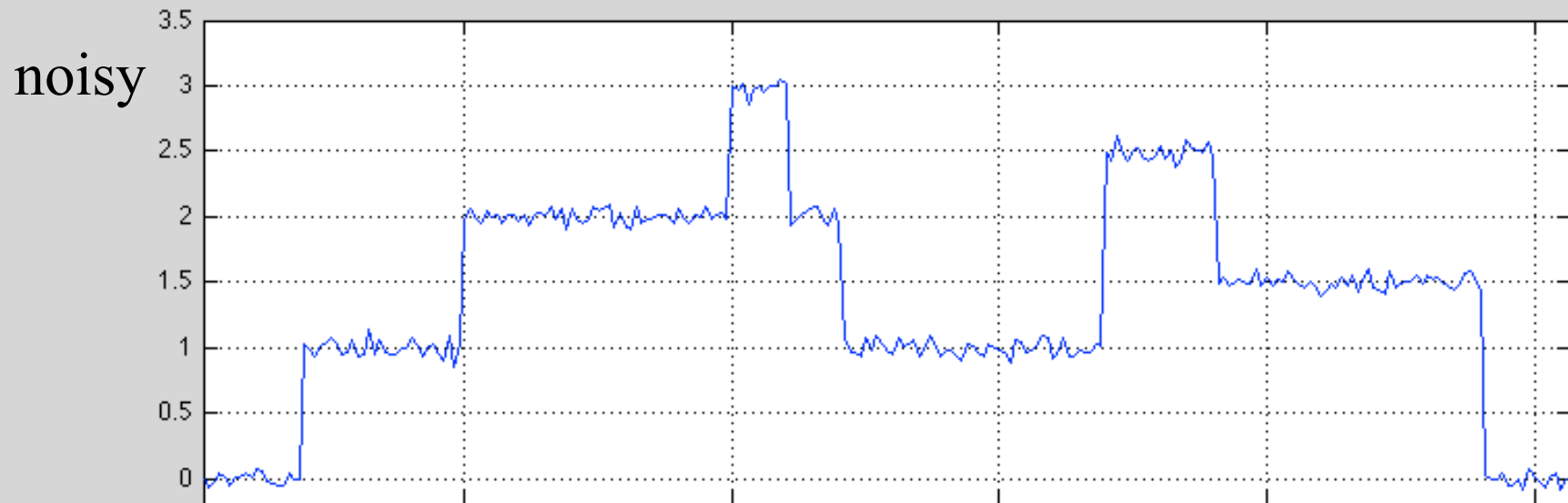
Example: Denoising Noisy Signals



Haar



Example: Denoising by Thresholding



Compression - JPEG2000 vs JPEG

Jpeg2000 - Wavelet



Jpeg - DCT



@ 66 fold compression ratio

Compression - JPEG2000 vs JPEG

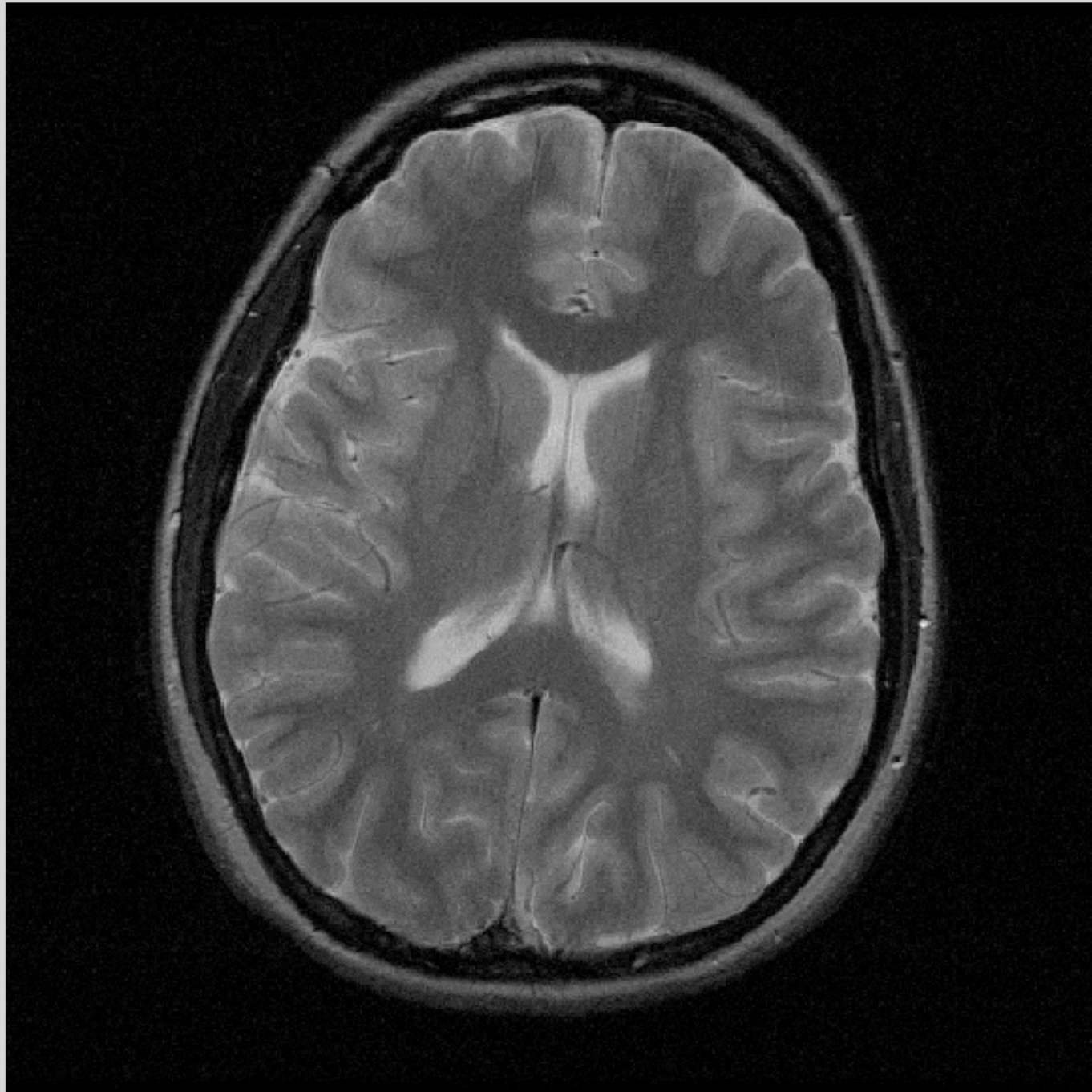
Jpeg2000 - Wavelet

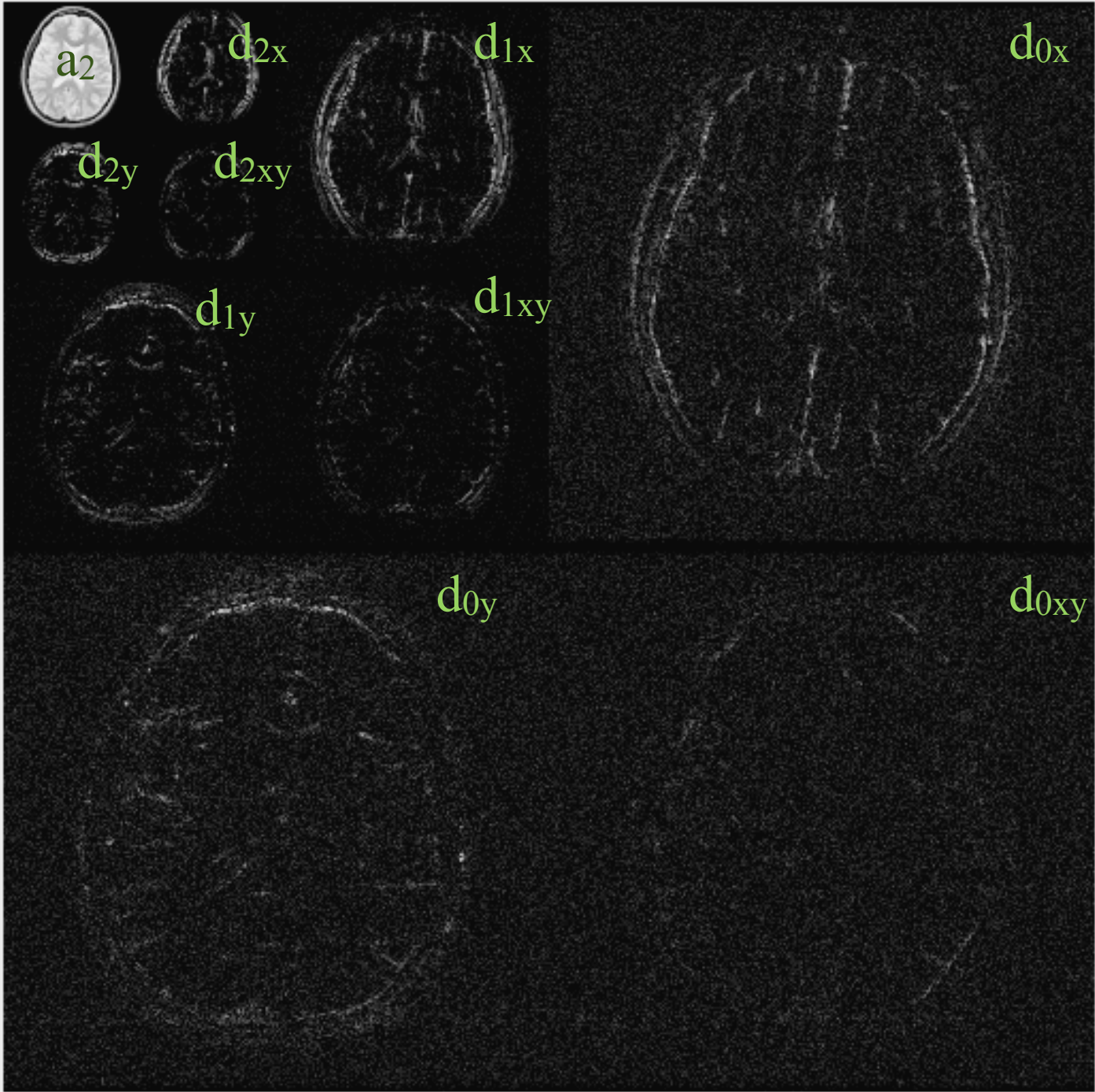


Jpeg - DCT



@ 66 fold compression ratio

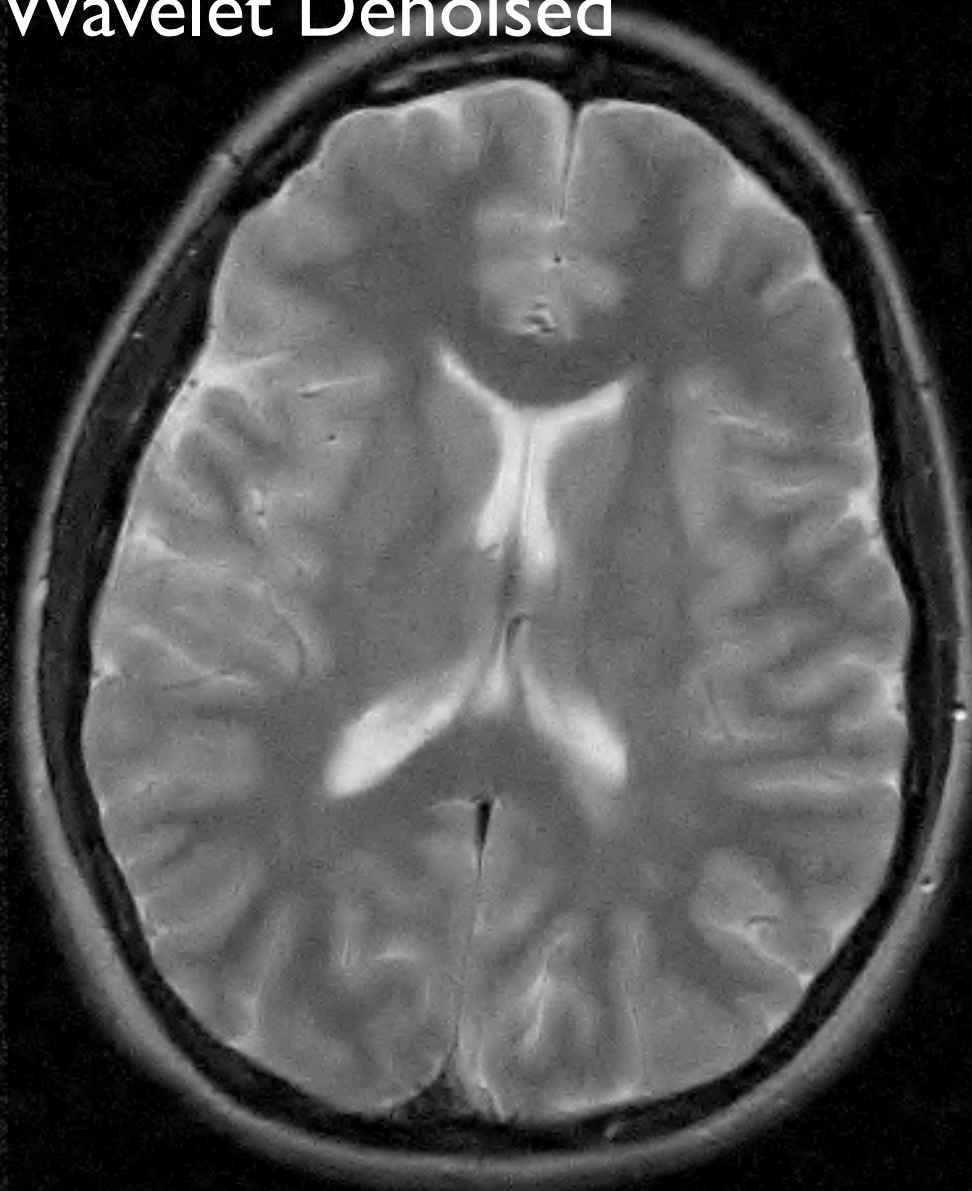




Noisy



Wavelet Denoised

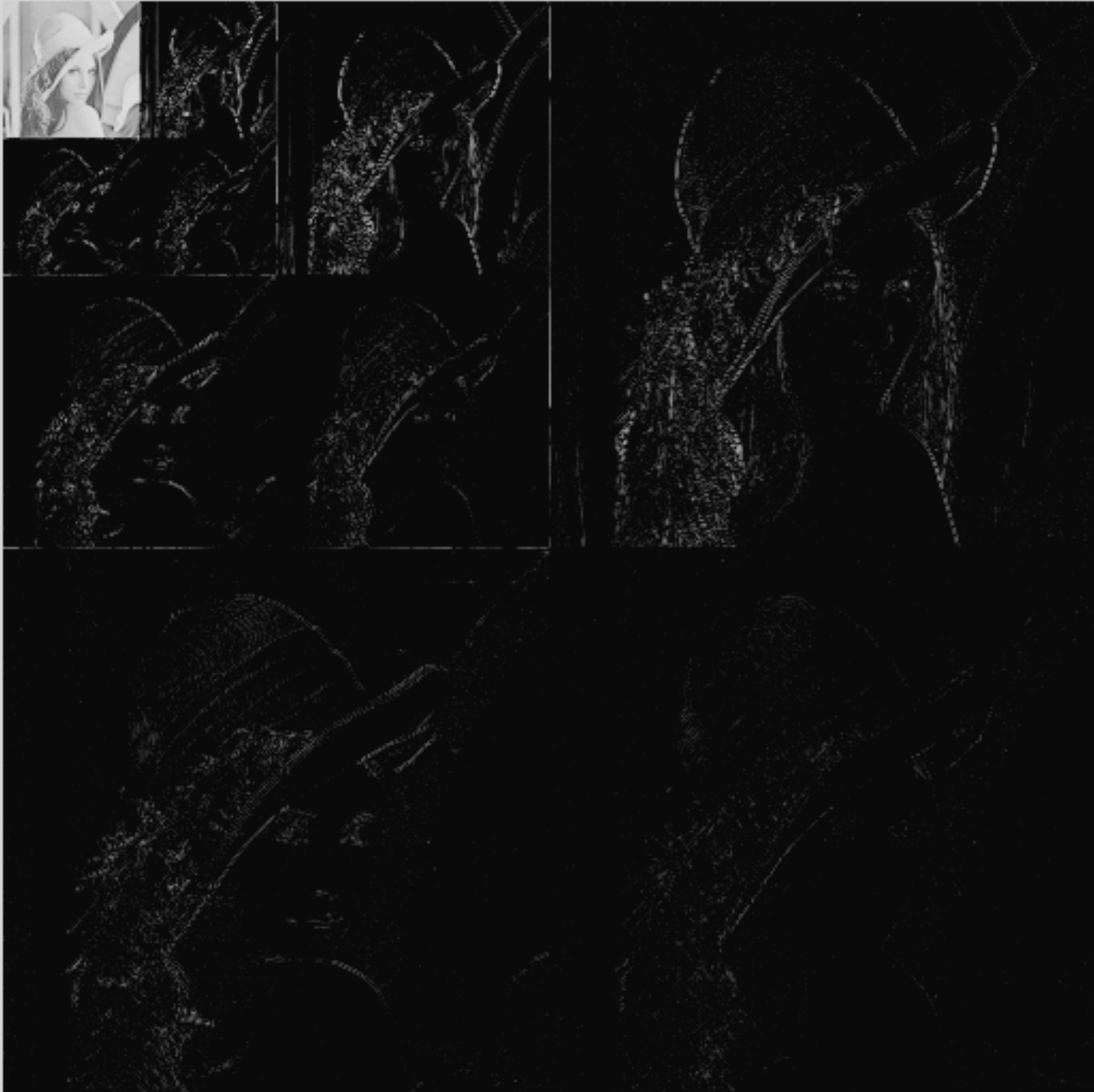


Approximation/Compression

0.000 % coefficients



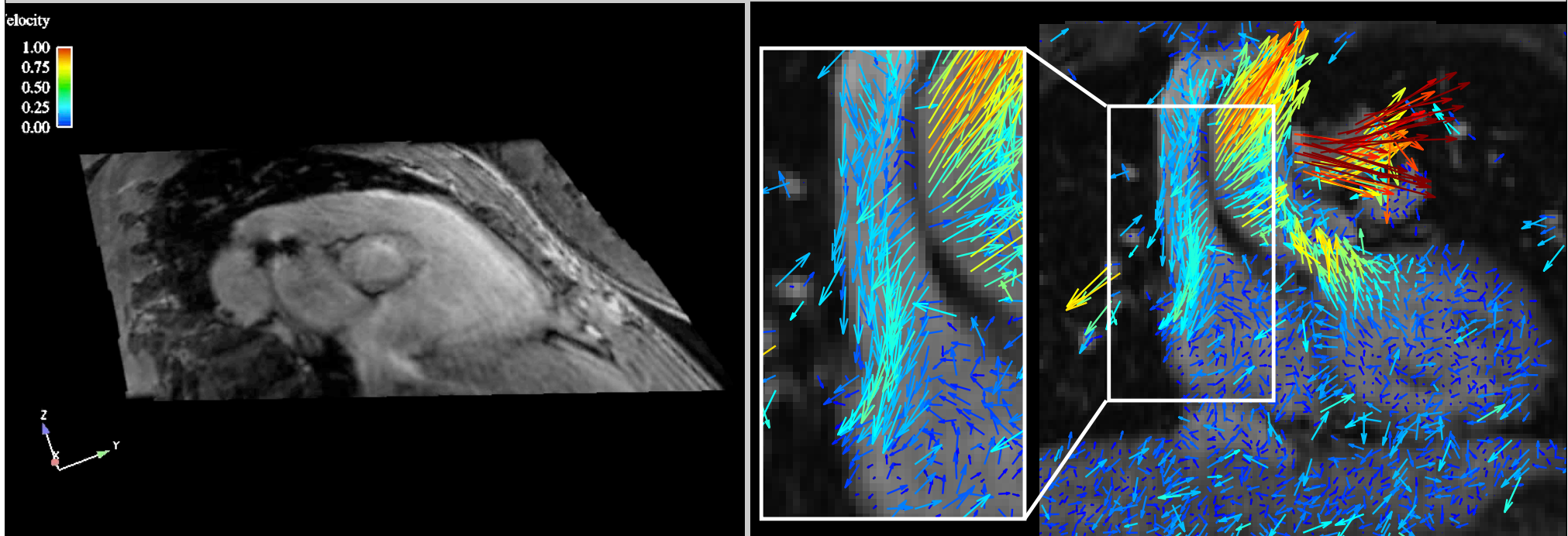




Example in Research

Robust 4D Flow Denoising using Divergence-free Wavelet Transform

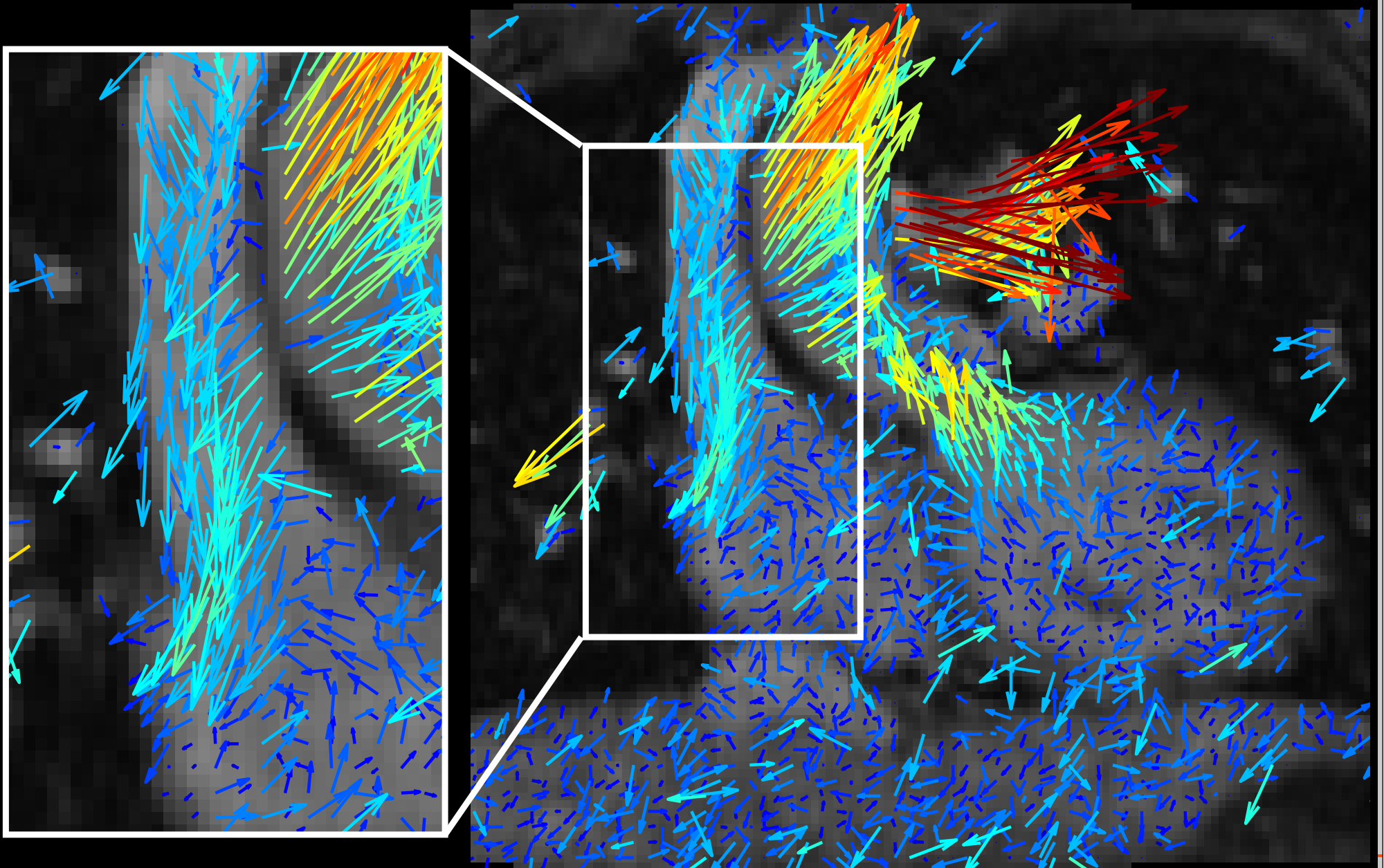
Frank Ong¹, Martin Uecker¹, Umar Tariq², Albert Hsiao², Marcus T Alley²,
Shreyas S Vasanawala², Michael Lustig¹



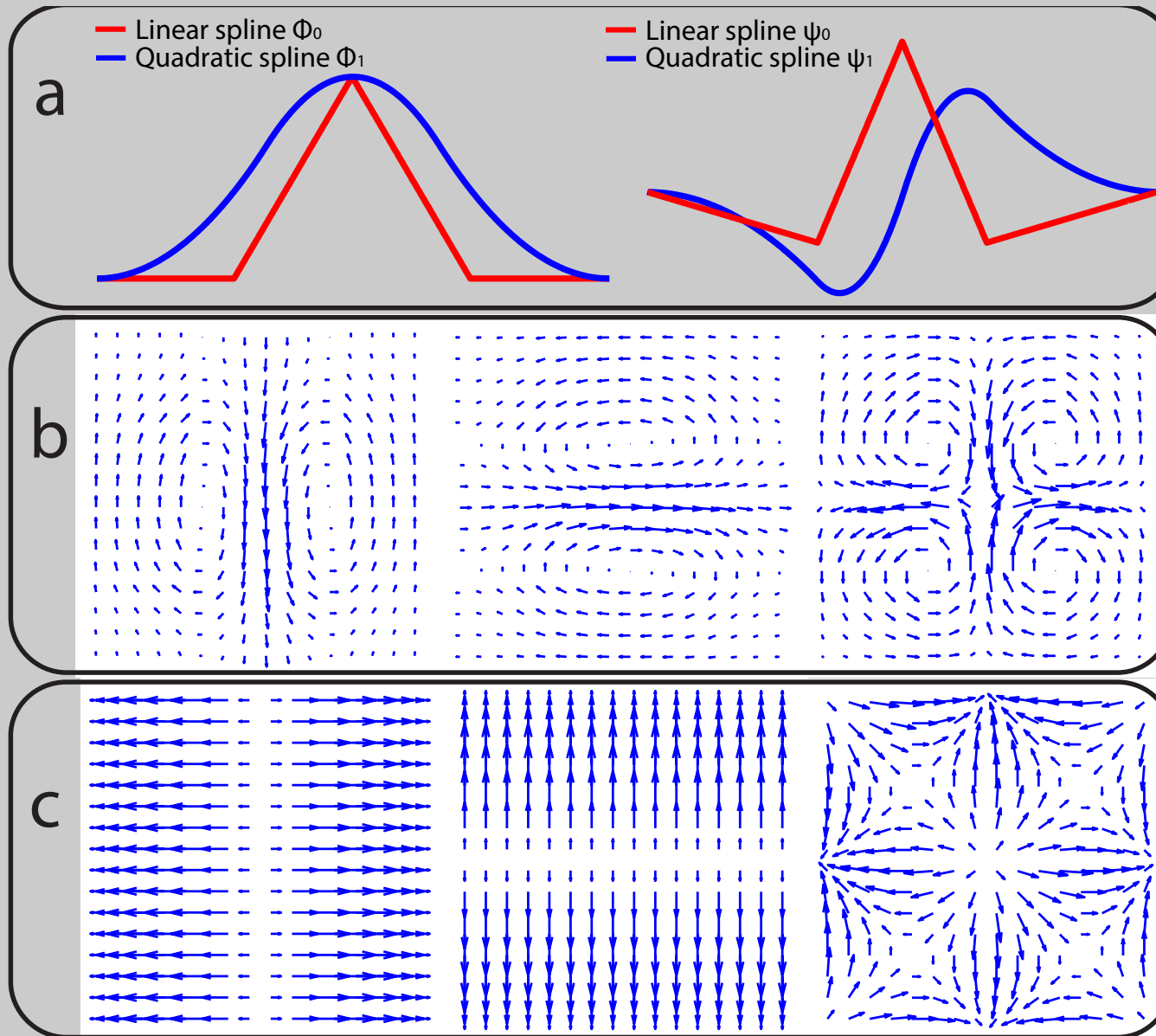
courtesy, Frank Ong and Marcus Alley

M. Lustig, EECS UC Berkeley

Noisy Flow Data



Divergence Free Wavelets



Divergence-Free Wavelet Denoising

