EE123
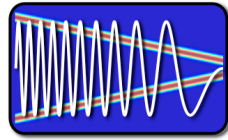# Digital Signal Processing

Lecture 15

Topics

- Last time
  - Ideal Sampling model   C/D
  - Impulse sampling $x_c(t) \Rightarrow x_s(t)$
  - Impulses to discrete samples $x_s(nT) \Rightarrow x[n]$
  - Relationship $X_c(j\Omega) \Leftrightarrow X_s(j\Omega) \Leftrightarrow X(e^{j\omega})$
  - Ideal reconstruction D/C
- Today
  - D.T processing of C.T signals
  - C.T processing of D.T signals (ha?????)
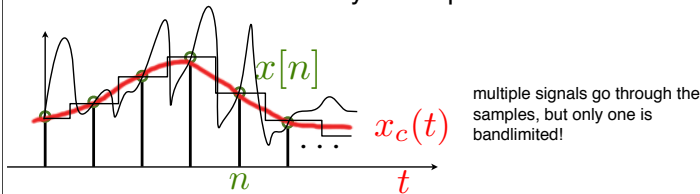 - Downsampling

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Reconstruction of Bandlimited Signals

- Nyquist Sampling Thm: suppose $x_c(t)$ is bandlimited

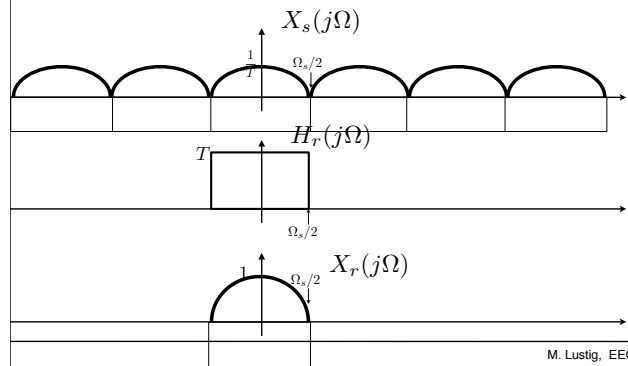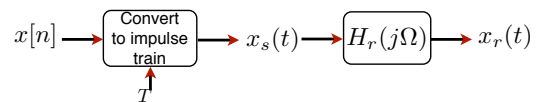$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

if $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

- Bandlimitedness is the key to uniqueness

multiple signals go through the samples, but only one is bandlimited!

Reconstruction in Frequency Domain

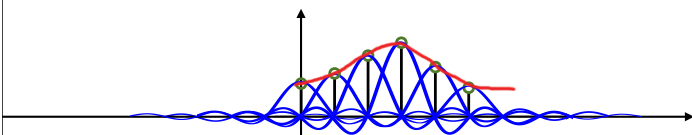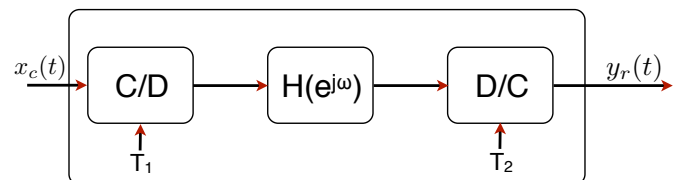## Reconstruction in Time Domain

$$
\begin{aligned}
x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t) \\
&= \sum_n x[n]h(t - nT)
\end{aligned}
$$

The sum of "sincs gives $x_r(t) \Rightarrow$ Unique signal bandlimited by Ω_s

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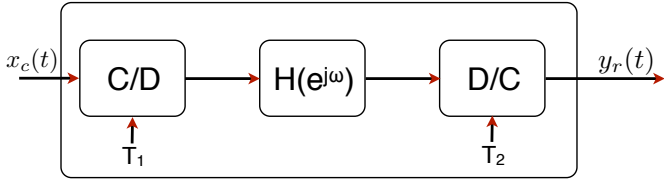## Discrete-Time Processing of C-T Signals

- Q: If $h[n]$ is LTI, H(e^{jω}) exists, Is the whole system LTI?

Slide 1

Discrete-Time Processing of C-T Signals

$x_c(t)$ → [C/D] → [H(e^{jω})] → [D/C] → $y_r(t)$

T_1 (under C/D), T_2 (under D/C)
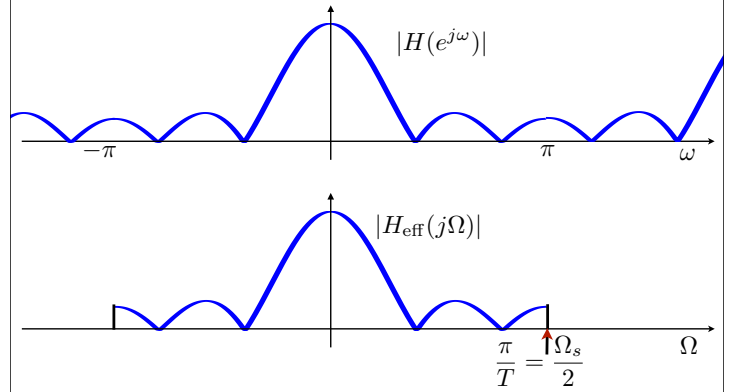
- Q: If *h[n]* is LTI, H(e^{jω}) exists, Is the whole system LTI?
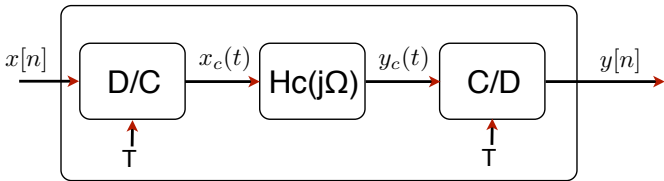- A: If xc(t) is bandlimited by $\dfrac{\Omega_s}{2} = \dfrac{\pi}{T}$ then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Slide 2

Example:

- Length 5 moving average

$|H(e^{j\omega})|$

$|H_{\text{eff}}(j\Omega)|$

$\dfrac{\pi}{T} = \dfrac{\Omega_s}{2}$

Slide 3

C.T Processing of D.T Signals

$x[n]$ → [D/C] → $x_c(t)$ → [Hc(jΩ)] → $y_c(t)$ → [C/D] → $y[n]$

T (under D/C), T (under C/D)

LTI - has H(e^{jw})!

- Useful to interpret D.T. systems with no simple interpretation in discrete domain.

- <u>Tool</u>: recall: $x_c(t) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]\,\text{sinc}\left(\frac{t-nT}{T}\right)$

Slide 4

Derivation

$$X_c(j\Omega) = \begin{cases} TX(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \quad \Rightarrow \text{also bandlimited}$$

so,

$$Y(e^{j\omega}) = \frac{1}{T}\sum_k Y_c(j(\Omega - k\Omega_s))\bigg|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}Y_c(j\Omega)\bigg|_{\Omega=\frac{\omega}{T}}$$
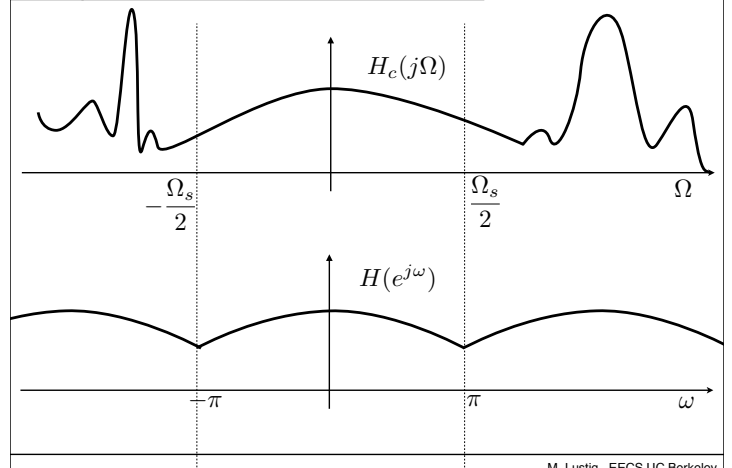
no aliasing!

Slide 5

Derivation

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_k Y_c(j(\Omega - k\Omega_s))\bigg|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}Y_c(j\Omega)\bigg|_{\Omega=\frac{\omega}{T}}$$

Combining the result:

$$Y(e^{j\omega}) = \underbrace{H_c(j\Omega)\big|_{\Omega=\frac{\omega}{T}}}_{H(e^{j\omega})} X(e^{j\omega}) \qquad |\omega| < \pi$$
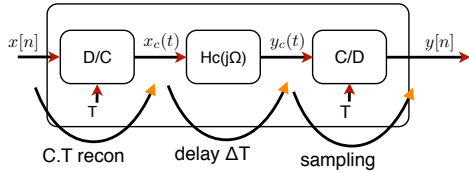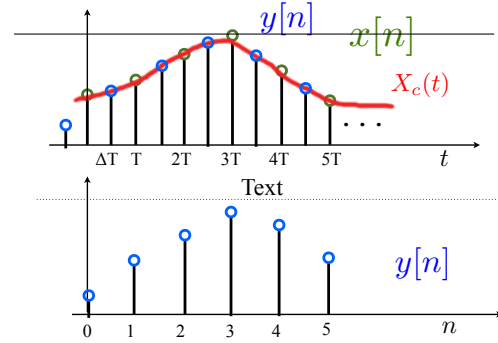
Slide 6

Example:

$H_c(j\Omega)$

$-\dfrac{\Omega_s}{2}$, $\dfrac{\Omega_s}{2}$

$H(e^{j\omega})$

$-\pi$, π

Example:

Non-integer delay: $H(e^{j\omega}) = e^{-j\omega\Delta}$

- What is the time-domain operation when Δ is not an integer ($\Delta=1/2$)?

Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ _delay of ΔT in time_

x[n] → D/C → $x_c(t)$ → Hc(jΩ) → $y_c(t)$ → C/D → y[n]

C.T recon   delay ΔT sampling

Example: Non Integer Delay

Text

Example: Non Integer Delay

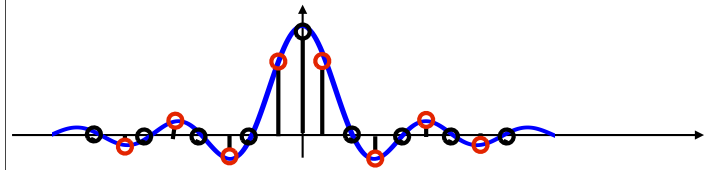- The block diagram is only for interpretation!

$$y_c(t) = x_c(t - \Delta)$$

$$
\begin{aligned}
y[n] &= y_c(nT) = x_c(nT - T\Delta) \\
&= \sum_k x[k]\,\text{sinc}\left(\frac{t - kT - T\Delta}{T}\right)\Bigg|_{t=nT}
\end{aligned}
$$

T's cancel!

$$= \sum_k x[k]\,\text{sinc}(n - k - \Delta)$$

Example: Non Integer Delay

$$h[n] = \text{sinc}(n - \Delta)$$

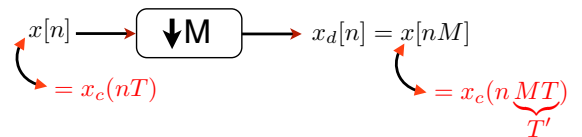Example: a discrete delta is a representation of a sampled sinc

shifted by partial samples results in many coefficients!

DownSampling

- Much like C/D conversion
- Expect similar effects:
  - Aliasing
  - mitigate by antialiasing filter

- Finely sampled signal $\Rightarrow$ almost continuous
  - Downsample in that case is like sampling!

Changing Sampling-rate via D.T Processing

Downsampling:

$x[n]$ → ↓M → $x_d[n] = x[nM]$

$= x_c(nT)$ $= x_c(n\underbrace{MT}_{T'})$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

Changing Sampling-rate via D.T Processing

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s} k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

we would like to bypass X_c and go from $X(e^{j\omega}) \Rightarrow X_d(e^{j\omega})$

substitute $r = kM + i$ $i=0,1,...,M-1$
$\quad\quad\quad\quad\quad\quad\quad\quad k=-\infty,...,\infty$

two counters
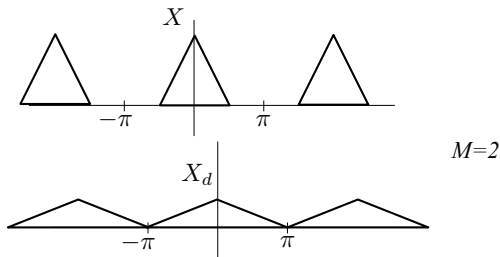*e.g.*, k: hours, i: minutes

Changing Sampling-rate via D.T Processing

$$
\begin{aligned}
X_d(e^{j\omega}) &= \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\underbrace{\frac{1}{T}\sum_{-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}k\right)\right)}_{X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})}
\end{aligned}
$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$$
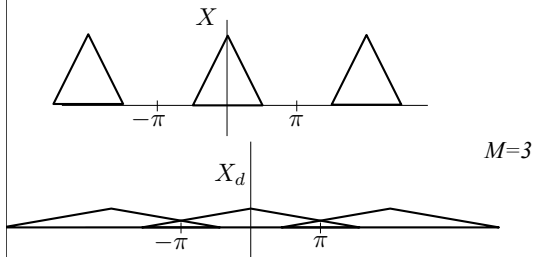
stretch   replicate
by M

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M - 2\pi i/M)}\right)$$

$M=2$

Changing Sampling-rate via D.T Processing

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0} X\left(e^{j(w/M - 2\pi i/M)}\right)$$

$M=3$

Anti-Aliasing

$M=3$